

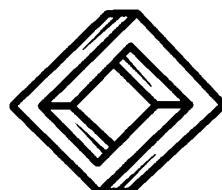
**Publicaciones Electrónicas  
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**Memoirs of the  
Fourth International Seminar  
on  
Mathematical Music Theory**

**Emilio Lluis-Puebla  
Octavio A. Agustín-Aquino  
(Editors)**

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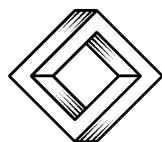




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## Preface

The Fourth International Seminar on Mathematical Music Theory took place in Huatulco, in the Mexican state of Oaxaca, from the 22nd to the 24th of November of 2010.

We consider it a great success, since many music-mathematics and mathematics-music professionals joined in a wonderful atmosphere to talk about Mathematical Music Theory. The First International Seminar on Mathematical Music Theory took place in Mexico City a decade before (in the year 2000), and the corresponding memoirs were firstly published in the SMM Memoir Series and were later republished together with the 2nd and 3rd International Seminars in Sauen (2001) and Zurich (2002) by Osnabrück University.

We asked Guerino Mazzola to update his view on the present state of the discipline, which is his *Mathematical Music Theory: Status Quo 2010*. In fact, it is the third of his 10-year-periodic status quo reports on Mathematical Music Theory (the previous one was also presented in México). As a particularity, it provides a time table indicating the increased success of Mathematical Music Theory in terms of the international societies, journals and corresponding conferences. It also gives an overview of “gestures”, which he proposes as the shaping force of future’s mathematical musicology. This paper is the first article of the Memoirs, since we think it is also an excellent introduction to the central themes of the discipline.

The next paper is *On D. Tymoczko’s critique of Mazzola’s counterpoint theory* by Octavio A. Agustín-Aquino and Guerino Mazzola. This paper replies some of Dmitri Tymoczko’s statements and questions concerning the counterpoint theory developed by the second author.

In *Computers with personality: a mathematical approach to musical style in improvisation* by Campos, Escamilla, Garay, Lobato, Mireles, Padilla, Paz, Rojas and Sánchez the notions of musical style and personality from a formal perspective are discussed. First they recall some important historical facts, from early music theorists to contemporary approaches and then they present an overview of mathematical tools that might be useful in developing a theoretical framework including formal languages, Markov chains, genetic algorithms, neural networks and finite state machines. At the same time they develop some algorithms illustrating the main ideas. This also includes a program based on finite state machines improvises in real time, incorporating some basic stylistic trends of some input fragments.

The next article is *Expanding compositional archetypes in a composer assistant tool* by Roberto Morales-Manzanares. In this paper he presents the configuration of a composition tool, which involves finite state machines, group theory using cyclic groups and dynamic systems theory using chaotic functions. As an example he presents Lutosławski’s procedures and mappings expressed in the musical analysis and studies done by Steven Stucky, Martina Homma, Adrian Thomas and Charles Bodman Rae.

In *The RUBATO® Composer software concept for learning advanced mathematics* Mariana Montiel explains although RUBATO®Composer is designed

primarily as an aid for the composer and music theorist, it is also an excellent tool for learning sophisticated mathematical concepts. In particular, RUBATO®modules (the so-called *rubettes*) were developed in order to hide the mathematics and present interfaces that allows the user to carry out abstract operations through gestural mouse commands, based on mathematical gesture theory. These high level rubettes could be uncovered, so that the music student can really begin to assimilate the mathematics on which modern mathematical music theory builds its edifice. At the same time, the mathematics student can witness an implementation of hitherto abstract concepts, and the computer science student encounters category theory in the development of the software itself. This paper develops these ideas from a pedagogical, mathematical and philosophical perspective, and reports on the case of a student and his experience with RUBATO®Composer.

In *Some extraspecial (and almost extraspecial) musical transformation groups*, Robert Peck examines finite groups (and subgroups) of musical transformations that are isomorphic to various extraspecial and almost extraspecial groups. These groups incorporate various 4-cycles that correspond to the imaginary unit  $i$  and which intersect in a central involution. These transformation groups act on sets of pitch-classes in an octatonic collection, and in various triadic-transformational systems. He also apply his results to an analysis of a large-scale inversional gesture in the musical composition *Cantata Profana* by Béla Bartók.

*D1 Theory, MúSIIC-Win and some applications in Musical Analysis: Six pieces for piano by Arnold Schoenberg* is a paper (in Spanish) by Julio Estrada. His text offers the basic concepts of  $d1$  theory, on the combinatorial potential of pitch scales and duration scales. This theory is not related to a composition system, and does not propose an aesthetic theory in order to organize the musical materials; it simply serves as a tool for creative search, analysis, pedagogy, musicology and ethnomusicology. Its latest development is the computer program MúSIIC-Win, that allows dealing with the combinatorial potential of 3 to 24-term scales. The  $d1$  theory is particularly useful in musical analysis of scores based on modal, tonal, chromatic scales, and of macro or micro-intervals. This method is based on interval classes, a concept from which sequential or vertical interval groupings can be observed, as well as identities and their permutohedrons, thus providing precise information on the combinatorics which itself allows to understand the idea of musical exploration of duration and/or pitch scales, used in a musical work. The article concludes with the analysis of Schoenberg's 6 pieces for piano. This composition reveals, from different angles, an exploration of tonality and a gradual immersion into atonality.

The *Prelude in three parts and  $\phi$*  by Francisco Guillermo Herrera Armendia is based on the work of Charles Madden, and the algorithms studied by Madden are applied to compose a musical work, a prelude in three parts where the author tries to adjust the golden section in the first and second parts and deliberately does not apply it to the third.

At the end we added the *Book Presentation* by Emilio Lluis-Puebla and Mariana Montiel which took place during the Seminar. The book presented was *An introduction to Group Theory with applications to Mathematical Music Theory* written by Octavio Agustín-Aquino, Janine du Plessis, Emilio Lluis-Puebla and Mariana Montiel.

We thank the partial support of the Sociedad Matemática Mexicana applied exclusively to finance part of the accommodation of some students and participants.

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# Mathematical Music Theory: Status Quo 2010

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*To the memory of my wife  
Christina Mazzola (1945-2009)  
for her loving life  
blossoming in my work.*

## Abstract

In this third of my 10-year-periodic status quo reports on mathematical music theory, I shall focus on two pillars: (1) a properly mamutheoretical one, dealing with the emergent paradigm of factorization. It focuses on the generic question/problem of how to move from one musical object to another one. Factorization is central in mathematics, and prime factorization in fact sets the frame of dimensionality in pitch via Euler space constructions. But it also appears in word monoid constructions to understand scale architectures, and then in Fourier's additive factorization of periodic functions into sinusoidal components. Factorization is also the central topic of transformation theory in that musical objects are related to each other by transformational diagrams. The extremal factorization case is infinite factorization, i.e. an infinite number of factors, which is already present in Fourier's approach, but can be extended to continuous factorial parameters, and this is what gesture theory deals with. Pillar (2) deals with an important epistemological theme: the problem of relating theory and experiment in mathematical music theory. We want to give a critical review of the status quo with respect to experimental evidence of mamutheoretical investigations.

## 1 Introduction

This third MaMuTh (mathematical music theory) status quo report, ten years after the second [22], and twenty years after the first [20], is not only a report on technical developments, which are impressive, but one, which to my mind testifies a dramatic extension of mathematical methods and theoretical topics dedicated to understanding music. This extension relates to the introduction of gestural perspectives as opposed to formulaic approaches. In this sense the report will first focus on more traditional views in section 3 “Factorization with

Facts and Processes”, and then on the gestural view in section 4 “Factorization with Gestures”.

We have chosen the title of “factorization” since this is the unifying aspect of traditional processes as well as gestural dynamics in music, and it reflects main efforts in the research of the last decade. Whereas factorization traditionally refers to the decomposition of a function, or process,  $f$ , into a product, composition, concatenation of two factors  $f = f_2 \cdot f_1$ , factorization in gestures refers to a connection of the identity function with  $f$ , described by a continuous curve of intermediate states. The latter could be understood as an infinitely fine factorization.

But factorization is not only a technical term, it also signifies the deeper topic of what could be a conceptual atom or element in thinking (and making) music. Can music be conceived as a combination of elementary particles? Or is it infinitely deep in its building blocks? The paradigm of atomic systems has an interesting history that originates with Greek philosophy, Democritus arguing that matter is composed of smallest objects, the atoms ( $\alpha\text{-τομος}$  the un-cuttable). The theory has received a huge success in the modern atomic theory of matter, but atoms turned out to be everything else but uncuttable. Atoms are built from elementary particles: electrons, protons, and neutrons. But even those particles are not atomic, they are built from quarks and (possibly) leptoquarks. However fine we classify matter, these particles have even been redefined not as point-like objects, but as extended strings in string theory. So the atomic concept in physics is being questioned by a concept that is centered around a curve-like structure that we would also call gestures (see section 4 for the definition of a musical gesture).

The concluding section 5 “Theory and Experiments” is dedicated to a politically and epistemologically crucial theme, the problem of verification or falsification of MaMuTh models, for the future development of MaMuTh and its role in the global science of music.

Before getting off ground with those factorization topics, let us summarize the process that led to the extension of MaMuTh to gestural considerations. When I first applied Grothendieck’s algebraic geometry, Gabriel’s quiver theory of modules, and their underlying functorial paradigm, to questions of music theory in 1978, I felt and was considered as being incoherent between my scientific reflection under the formulaic compression of mathematics and the artistic search for sounding gestuality in my practice as a free jazz pianist. It took me years, in fact, until my talk at the IRCAM in Paris on May 18, 2002, where I tried in vain to explain my free jazz playing in terms of mathematical formulas that I had extensively described in my book “The Topos of Music” [23], to embark in a serious search for a coherent theory comprising formulas and gestures in music. That talk turned out to be a dramatic event since I realized that it was not mathematical formulas, which controlled and shaped my improvisation, but real gestures, utterances not driven by formulaic abstraction, simply dancing fingers, in summary: a totally different world of musical reality. I understood that music accepts formulas, compact representations of facts, but it cannot be realized except in a dynamic move away from those formulas towards unfolding

gestures, the making, the realization of the dynamical power of such formulas in musical performance, in other words: in embodied intelligence.

This movement from formulas to gestures turns out to be characteristic for an overarching comprehension of music. Music needs the score, whose processual organisation makes the musical formulas radiate into a plethora of interpretations, but the score also needs performance, where its casted gestures are resuscitated to living knowledge. In a categorical setup this vibration between music and mathematics, where music unfolds formulas into gestures, and mathematics wraps and compresses gestures to formulas would be described by adjoint functors between the category of formulas and the category of gestures:

$$\begin{array}{ccc} & \text{music} & \\ \text{formulas} & \longleftrightarrow & \text{gestures} \\ & \text{mathematics} & \end{array}$$

It was Alexander Grothendieck who made thoroughly plausible that gestural dynamics might play a major role in mathematics. On the level of academic education mathematics unfolds in theories, in operational schemes and processual diagrams. Nicolas Bourbaki's work is a prototype of this industrial level of processual embodiment. But making mathematics, the creation of conceptual bodies and associated theories, is radically different from those catechisms. Ives André, a leading French scholar in Grothendieckian algebraic geometry, has pointed out to me that on this level of creative mathematics, gestures are pivotal.

In the spirit of Grothendieck's philosophy of mathematical problem solving we have to message the conceptual skin, the interface between internal and external layers that define the problem and delimit it from known regions. This dialectic of successively and patiently softening the conceptual skin is famous from Grothendieck's nearly endless strings of conceptual propagation in his proofs until the soft movement of your hands makes the problem dissolve.

In section 4, we shall draw a gross image of the movement of mathematical music theory from its formulaic shape in the Fregean prison, as prominently exemplified by the classification theorem of musical manifolds, to the processual schematism of category theory, as forwarded by the so-called transformational paradigm in modern music theory, and finally to its gestural unfolding towards a mathematical theory of gestures, which not only meets the musical substance in the performative making, but also offers a reconstruction of gestural substance from abstract category theory.

Before delving into technicalities, I should express my gratitude to all my colleagues, collaborators, and friends who made it possible to develop a MaMuTh with international resonance. Above all, I should express my deepest gratitude to my wife Christina who passed away too early, however she could experience with great satisfaction that her support of my crazy ideas and initiatives has led to a full success of that MaMuTh enterprise we had jointly initiated more than 25 years ago.

## 2 Time Table

I think that in view of the remarkably successful recent history of MaMuTh, we should prepend a time table. It relates to theory, software, events, and music as mentioned in our three status quo reports for the years 1990, 2000, and 2010. The table's contents are a personal perspective and do not mean that what has been omitted is insignificant. The table however demonstrates the increased success of MaMuTh in terms of the international society, the journal, and corresponding conferences. I have not included events in 2011, in particular the third international MCM conference at the IRCAM, Paris, which will be a fascinating milestone for the next status quo report in 2020.

Year	Theory	Software	Grants	Music
1978				LP Kelvin Null
1980				LP Akroasis
1981	Gruppentheoretische Methode in der Musik	MDZ71 software		
1984	Karajan Symposium, Salzburg and Keynote talk at the Jahresversammlung of the Deutsche Mathematiker Vereinigung, Kaiserslautern			
1985	Gruppen und Kategorien in der Musik		Depth-EEG for Con- and Dissonances	
1988		presto®		
1990 Status quo I	Geometrie der Töne			CD Synthesis
1992			RUBATO® project	CD Immaculate Concept
1995	Thomas Noll: Morphologie abendländischer Harmonik			
1996		RUBATO® NeXT		Kuriöse Geschichte
1998		RUBATO® Mac OSX	KiT-MaMuTh project	
2000 Status quo II			1st Int. MaMuTh Seminar, Mexico City Medal of Mexican Math. Soc.	Joachim Stange-Elbe's Kunst der Fuge
2001			2nd Int. MaMuTh Seminar, Sauen 3rd Int. MaMuTh Seminar, Zürich	
2002	The Topos of Music my gesture shock	RUBATO® for Java	SNSF grants	
2003 2003	K-nets = Limits	Stefan Müller's EspressoRubette PianistHandRubette	Autour de la Set Theory IRCAM US-European Colloque	
2004	Powerset classification = Limit	Stefan Göller's Primavista Brower		
2006 2006 2006	Gérard Milmeister's RUBATO® Composer		MCM Society founded in Louisiana by Robert Peck	
2007	Math. Gesture Theory of Music	Florian Thalmann's BigBangRubette	1st International MCM Conference in Berlin Journal of Mathematics and Music (Noll & Peck)	Restructures from Boulez's Structures Ia
2008	1000 Topos of Music sold			
2009	Categorical Gestures		2nd International MCM Conference at Yale U	
2010 Status quo III			4th Int. MaMuTh Seminar, Huatulco	

## 3 Factorization with Facts and Processes

Since the metaphysical Pythagorean tetractys symbol, western harmony has been based on the idea of prime factorization of integers. The classical just tuning, extending Pythagorean tuning (relating to the primes 2 and 3) by the major third 5/4 (relating to the prime 5), comprises the frequencies  $f =$

$f_0 2^o 3^q 5^t$ ,  $o, q, t \in \mathbb{Q}$ , where  $f_0$  is a base frequency. In terms of pitch, the logarithm of frequency, we therefore consider the *Euler space*  $E$  [8] of rational linear combinations  $o \log(2/1) + q \log(3/2) + t \log(5/4)$ , which is a three-dimensional  $\mathbb{Q}$ -vector space by the unique prime number decomposition of integers. This linear factorization of pitches defines a kind of pitch “atoms”, the basis  $\log(2/1), \log(3/2), \log(5/4)$  given by the musically meaningful octave, fifth, and major third. Classical harmony refers to this vector space’s basis.

More recently, the semigroup **Aff** of all affine endomorphisms of the space of onset, pitch, loudness, and duration, has been investigated for its possible musical signification. It can be shown that every transformation in **Aff** is a composition of a small number of generators which are all musically significant, moreover it is possible to exhibit a set of generators which are each only affecting two parameters [23, p.159, Theorem 2]. The typical factorization here is that the rotation of  $180^\circ$  in the onset-pitch plane is a product of pitch inversion and retrograde, both being musically significant. In music theory, their product is called “retrograde-inversion” which shows that in music, the rotation is only significant *qua* product of those classical operations. This principle has been called “concatenation principle” in [19], meaning that a transformation is musically meaningful iff it can be written as product of musically meaningful factors. The difference to the Euler space factorization is that here, we are not concerned with unicity, but only existence of a set of generators that are musically meaningful.

This approach has also been followed by Thomas Noll’s investigation of the musical semantics of generators of  $SL(2, \mathbb{Z})$  [34], since he then relates this group to the pitch class group by the isomorphism

$$SL(2, \mathbb{Z}) / (SL(2, \mathbb{Z}), SL(2, \mathbb{Z})) \xrightarrow{\sim} \mathbb{Z}_{12}.$$

He had already used this approach in the generalization of the author’s circle chords<sup>1</sup> to “bigeneric” chords, i.e. chords generated by two endomorphisms, in his dissertation [34].

Noll’s reconstruction of Riemann’s harmony using bigeneric chords, as well as our work on the concept framework for musical objects implemented in the software RUBATO® [21] had a deep consequence also in terms of the mathematical framework necessary to understand music. Let us first discuss this before we expose a far-reaching factorization, namely the local-global dichotomy in music.

In the first years of the last decade of the 20th century, mathematical music theory was governed by two perspectives: linear algebra and set theory. But these theories are quite incompatible in the sense that (iterated) powersets are not natural objects for linear algebra, and of course vice versa: Linear algebra is not a natural layer in set theory (although it can be constructed from set theory). Whereas linear algebra hosts musical notes as points in adequate modules, it does not offer (iterated) powersets of notes as linear concepts. In terms of topos theory, this means: the category of modules has no subobject classifier  $\Omega$  and

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<sup>1</sup>Subsets of  $\mathbb{Z}_{12}$  defined as orbits of one pitch class under the monoid generated by one endomorphism of  $\mathbb{Z}_{12}$ , see below 3.4.1 for more details.

power objects  $\Omega^X$ . On the other hand, the category **Sets** of sets is a topos and therefore has a subobject classifier  $\Omega = 2 = \{0, 1\}$  and power objects  $Y^X$  (in particular powersets  $\Omega^X$ ), but it is completely unalgebraic, and therefore, neither the category of modules nor the category of sets are sufficiently rich to encompass both, symmetries and powersets.

At the end of the 20th century, the efforts of the author's working group joined at the most powerful mathematical language: topos theory, for historical, theoretical, and technological reasons; the historical reason relates to Wolfgang Graeser's work and is discussed in [32], we come back to the theoretical and technological reasons below. This theory was developed from sheaf-theoretic algebraic geometry by Grothendieck, following ideas from Jean Leray and Henry Cartan, and complemented in its logical perspectives by Charles Ehresmann, William Lawvere, and Myles Tierney. It is remarkable that topos theory had already been proposed prior to its musical application as a foundation of theoretical computer science by Dana Scott, but it is even more remarkable that topos theory seems to have not been proposed as a foundational approach to theoretical physics until very recently by John Baez, Andreas Döring, Chris Isham and others, and certainly after its application to music by the present author.

The minimal solution of a category<sup>2</sup> expressing both, subobject classifiers, and power objects as well as (affine) algebraic structures is the topos of presheaves **Mod**<sup>®</sup> over the category **Mod** of modules and di-affine morphisms. Its objects are the set-valued contravariant functors (the presheaves)  $F : \mathbf{Mod} \rightarrow \mathbf{Sets}$ , and its morphisms are the natural transformations between such presheaves.

The *theoretical* reason for the choice of the topos **Mod**<sup>®</sup> relates to Yoneda's lemma. We not only want to work in a topos, but also to relate the classical approach to the topos without losing information. Yoneda's lemma does exactly this job: The Yoneda embedding  $\mathbf{Mod} \rightarrow \mathbf{Mod}^{\circledast} : M \mapsto \text{Hom}(-, M) = @M$  is fully faithful, so two modules are isomorphic iff their associated presheaves are so. More precisely, if  $F$  is a presheaf and  $A$  is a module, then we have a bijection  $\text{Hom}(@A, F) \xrightarrow{\sim} F(A)$ . We therefore also write  $A@F$  for  $F(A)$ . Following Grothendieck, we call a morphism  $p : @A \rightarrow F$ , i.e. an element  $p \in A@F$ , an  $A$ -addressed *point* of  $F$ . The lemma then means that knowing about all point sets  $\text{Hom}(A, M) = A@M$  and their set maps  $@f : B@M \rightarrow A@M$  for di-affine morphisms  $f : A \rightarrow B$  determines all of the module  $M$ . For example, the underlying set  $M$  is in bijection with the set  $@M(0) = 0@M$  of 0-addressed points of  $M$  (i.e. of the associated presheaf  $@M$ ). We shall come back to this setup and give concrete examples from music theory to illustrate the power of this language.

Let us also give the *technological* reason for switching to topoi. Music is all but a purely intellectual game, whenever you want to prove something as being really relevant to music, you have to make music at some point, you have to make an experiment, much as theoretical physics would not be more than a *Glasperlenspiel* without experimental tests. We come back to this topic in

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<sup>2</sup>All categories in this discussion are supposed to be locally small.

section 5. In music research such experiments are canonically tied to computerized simulation of composition, analysis, or performance. In the software the author and his collaborators have developed (*presto*<sup>®</sup> for composition [23, ch.49], and RUBATO<sup>®</sup> for composition, analysis, and performance [23, part X], [21]) in the last thirty years, conceptual principles, i.e. the adequate format for music objects, represented in corresponding object-oriented classes in Objective-C or Java, became more and more important, since the interaction of software components was heavily dependent upon exchangeability of objects.

To achieve this goal, the language of *denotators* and *forms* was developed. Without describing the technical details of this language it can be said that it splits into an architectural framework and a mathematical core format. It is the latter that explicitly refers to a topos, namely the topos **Mod**<sup>®</sup>. This has been successfully implemented by Gérard Milmeister [33]. The toposic features used for programming music objects are those powerset objects mentioned above, on the one hand, and then objects related to limits and colimits, on the other. More concretely, limits are omnipresent in programming environments since they represent lists, arrays, vectors, and the like. But from the theoretical point of view, we were more focused on powerset constructions, since the classical objects of music theory are sets of notes (like chords, or melodic motives) in adequate parameter modules, or else sets of such sets, etc. We shall come back to the music related limit and colimit objects in a topos in the next section.

### 3.1 Functors for Fux and Riemann

For the time being, we concentrate on the powerset objects in music theory. Let us show how already harmless objects generate more and more functorial ones. Take the most common powerset objects, chords  $V \subset \mathbb{Z}_{12}$  of pitch classes. In our functorial setup, such a chord is a set of zero-addressed points, i.e.,  $V \subset 0@\mathbb{Z}_{12}$  in the representable functor  $@\mathbb{Z}_{12}$  given by Yoneda's construction. For example, we have the tonic triad  $Tc = \{0, 4, 7\}$  and the dominant triad  $Dt = \{7, 11, 2\}$  in C-major (the pitch class of tone  $c$  being represented by 0, etc.).

In Western music theory, in fact since the early days of Pythagorean interval categories, harmony has been the dominant issue. It deals with the rules of syntactical chord combinations as related to their semantical loading. The most ambitious enterprise in harmony was Hugo Riemann's attempt to attribute to every possible chord  $Ch$  (sic!) a semantic loading that reflects the chord's relation to a given tonality  $X$ , namely one of the three values,  $T$  (tonic),  $D$  (dominant), or  $S$  (subdominant), in order to express its role for a given tonality  $X$ . In fact, this attribution effectively *defines* tonality, i.e., we may write  $T = X(Ch)$ ,  $D = X(Ch)$ , or  $S = X(Ch)$  and view  $X$  as a function  $X : Ch \mapsto T, D, S$  on the set of chords. This project was however never completed. Only simple chords (such as major or minor triads and other standard chords) have been attributed such values, whereas the would-be harmonic logic seems to break down for different reasons, even for those simple chords. See Carl Dahlhaus' illuminating discussion [6], our comments in [23, ch.13.4.2.1], and also our hint in section 3.3 to an important reason of that failure, namely the non-orientability of the

harmonic band. This outage has been one of the main rationales for the dissociation of modern music theory from harmonic principles, as typically traced in Arnold Schönberg’s emancipation of dissonance in dodecaphonism.

In his successful (re)construction of harmonic logic according to Riemann’s ideas, Thomas Noll [34] had to consider more general chords, namely the monoids  $\text{Trans}(V, W) = \langle f \in \mathbb{Z}_{12} @ \mathbb{Z}_{12} | f(V) \subset W \rangle$  generated by the affine morphisms  $f \in \mathbb{Z}_{12} @ \mathbb{Z}_{12}$  mapping  $V$  into  $W$ . Such transporter sets are not normal zero-addressed chords, but  $\mathbb{Z}_{12}$ -*i.e.* self-addressed. Therefore the musical setup requires more general addresses than the naive zero address. The transporter  $\text{Trans}(Dt, Tc)$  was understood as the set of *Riemannian relative consonances* from  $Dt$  to  $Tc$  in Noll’s harmony. This set shows the following remarkable relation to classical counterpoint as modeled by Johann Joseph Fux [9] in 1725.

In contrast to harmony, counterpoint stresses on the composition of several melodic voice lines. Essentially, it defines rules determining which notes of the basic *cantus firmus* line can be played against which notes of a second line, the so-called *discantus*. The Fux theory is built upon the most basic case, called *first species* or *note-against-note*. This means that one is given a *cantus firmus* melody (built following the building rules from Gregorian chant), and that one has to add to every *cantus firmus* note a note of the *discantus* line according to a number of rules. The most important rule is that the intervals between *cantus firmus* and corresponding *discantus* notes must be consonant, *i.e.* modulo octave elements of the set of consonant interval quantities  $cs = \{0, 3, 4, 7, 8, 9\} \subset \mathbb{Z}_{12}$ . The second strong rule forbids the immediate succession of fifths (interval 7), *i.e.* “parallels of fifths” are forbidden. However, the rationales for the counterpoint rules are not very transparent. In particular, the parallels of fifths are also thought to be annoying (a psychological argument). In order to better understand the structural basis of counterpoint rules, the first species has been modeled by use of dual numbers by the present author [23, part VII]. It allows to deduce the rules for the note-against-note species, in particular the rule of forbidden parallels of fifths, from the unique affine isomorphism  $AK(x) = 5x + 2$  exchanging the set of consonant interval quantities  $cs$  and the complement, the dissonant quantities  $ds = \{1, 2, 5, 6, 10, 11\}$ . This model has the advantage that it allows for counterpoint compositions for interval dichotomies different from the classical consonant/dissonant case. Look at the set  $Cons = \{x + \varepsilon k | x \in \mathbb{Z}_{12}, k \in cs\}$  of consonant contrapuntal intervals represented as dual numbers with consonant infinitesimal component  $k$  and *cantus firmus*  $x$ . Consider the self-addressed chord  $\text{Trans}(Cons, Cons) = \langle f \in \mathbb{Z}_{12}[\varepsilon] @ \mathbb{Z}_{12}[\varepsilon] | f(Cons) \subset Cons \rangle$ . Then we have a canonical injection  $i : \mathbb{Z}_{12} @ \mathbb{Z}_{12} \rightarrow \mathbb{Z}_{12}[\varepsilon] @ \mathbb{Z}_{12}[\varepsilon]$ , and the relation states that

$$\text{Trans}(Dt, Tc) = \text{Trans}(Cons, Cons)|_i$$

This relation between Fuxian counterpoint and Riemannian harmony is still a miraculous fact, and its consequences are not yet investigated, much the same as the deeper relations between harmony and counterpoint are not understood to date and have given rise to a bunch of approaches subsumed under the title

of “voice leading” theory.

$A$ -addressed subsets  $C \subset A@M$  of modules can be seen as points  $C \in A@2^{\otimes M}$ , where by definition,  $A@2^{\otimes M} = 2^{A@M}$  is the evaluation of the functor  $2^{\otimes M}$  at address  $A$ . This functor is not representable, and hence, we are automatically concerned with non-representable presheaves. Which means that iterated powerset constructions would ask for subsets in non-representable presheaves, such as a set of chords.

However, a subset is not a presheaf, and therefore, one introduces for each subset  $C \subset A@F$  of  $A$ -addressed points of any presheaf  $F$  an associated sub-presheaf  $\widehat{C} \subset @A \times F$ . It is defined at the address  $B$  by  $B@\widehat{C} = \{(f : B \rightarrow A, c.f) | c \in C\}$ . This is a powerobject construction, since the sub-presheaves of  $@A \times F$  are precisely the elements of  $A@\Omega^F$ . It has been systematically applied in a topological theory of harmony, as exposed in [23, ch.24.2]. In other words: the functorial setup is not a general mathematical nonsense, but meets the reality of modern mathematical music theory.

This functorial setup generates a category  $Loc$  of local powerset structures, called *local compositions*, as follows: The category’s objects (the local compositions) are the subfunctors  $K \subset @A \times F$ , i.e., elements of  $A@\Omega^F$ . We call  $A$  the address and  $F$  the ambient space of  $K$ . Given two such local compositions  $K \subset @A \times F$ ,  $L \subset @B \times G$ , a morphism  $f/\alpha : K \rightarrow L$  is a pair consisting of a natural transformation  $f : K \rightarrow L$  and a module morphism  $\alpha : A \rightarrow B$  such that there is a natural transformation  $h : F \rightarrow G$  that makes the following diagram commute:

$$\begin{array}{ccc} K & \xrightarrow{\text{inclusion}} & @A \times F \\ f \downarrow & & \downarrow @\alpha \times h \\ L & \xrightarrow{\text{inclusion}} & @B \times G \end{array}$$

This category contains the full subcategory  $ObLoc$  of *objective* local compositions  $K = \widehat{C}$  and the subcategory of  $LocMod$  of local objective compositions  $C \subset A@M$  with representable (modular) ambient spaces  $F = @M$ . If everything is restricted to the category  $RMod$  of modules over a fixed commutative ring  $R$  (with  $R$ -affine morphisms), we obtain the categories  $RLocMod \subset RObLoc$  of local objective, or local  $R$ -modular compositions. Categories of local compositions have been classified for a number of important cases, comprising all (zero- or self-addressed) chords of pitch classes or motives in pitch and onset class spaces, see [23, ch.11.3]. We shall not discuss classification of local compositions here since this is a special case of the classification of global manifolds in music, called *global compositions*, which we shall discuss extensively later in this section.

### 3.2 Address Changes for Boulez’s *Structures pour deux pianos*

General addresses abound in music, and so do the address changes  $\alpha : A \rightarrow B$ , which define functorial transitions  $B@F \rightarrow A@F$ . For example, a dodeca-

phonic series  $S$  can be modeled as a  $\mathbb{Z}^{11}$ -addressed point of  $\mathbb{Z}_{12}$ . The retrograde series is then just the series  $S.r$ , where  $r$  is the address change on  $\mathbb{Z}^{11}$  defined by the permutation  $r(e_i) = e_{11-i}$  of the canonical affine base  $e_0 = (0, 0, \dots, 0)$ ,  $e_1 = (1, 0, \dots, 0)$ , ...,  $e_{11} = (0, 0, \dots, 0, 1)$ . Functorial address change is not merely a academic exercise, but allows to understand most advanced compositional strategies, as, for example, the serial approaches as set forth by Pierre Boulez in his famous *Structures pour deux pianos*. We have described this in detail in [27]. We shall just give a short sketch to illustrate our claim, which is—let us stress this—the naturality of the functorial point of view in music theory, analysis, composition, and performance.

The original problem of Boulez's *Structures* was that the serial data, namely a series  $S_P \in \mathbb{Z}^{11}@P$  for each parameter space  $P$  for pitch classes, onsets, durations, loudness, and attacks, cannot be handled uniformly when applying classical dodecaphonic transformations. It might be reasonable to apply an inversion on pitch classes, but such an operation makes no sense on attacks. The serial principle creates a problem not on the series, but on its transformations. More precisely, how would one transfer inversion, for example, from pitch classes to attacks? Boulez's answer is as modern as simple! He (implicitly) observes that any symmetry  $f$  on the pitch class space  $\mathbb{Z}_{12}$  that is used to create from the pitch class series  $S_{pitch} : \mathbb{Z}^{11} \rightarrow \mathbb{Z}_{12}$  a transformed series  $f \circ S_{pitch}$ , can also create that transformed series by an address change  $C_f : \mathbb{Z}^{11} \rightarrow \mathbb{Z}^{11}$  such that  $f \circ S_{pitch} = S_{pitch}.C_f$  (but the converse is wrong). Therefore, transformed series can be realized by address change, not by transformations of their ambient spaces. Boulez's solution is now immediate: In state of looking for transformations on attack and other problematic parameter spaces, one just prepends the associated address change  $C_f$  to that series. So, for example, if  $S_{attack}$  is the attack series, the transformed attack series corresponding to the transformation  $f$  in pitch class space would be  $S_{attack}.C_f$ . With this trick, Boulez manages to construct an entire zoo of series in all these parameters by use of canonical address changes. He refers to a  $12 \times 12$ -matrix  $Q$  defined as follows: Its  $i$ th row  $Q(i, -)$  is the base change  $C(T^{S_P(i)-S_P(1)})$  associated with the transposition<sup>3</sup>  $T^{S_{pitch}(i)-S_{pitch}(1)}$  by the difference of the pitch class series  $S_{pitch}$  at position  $i$  and 1. This matrix is equivalent to an address change  $Q : \mathbb{Z}^{11} \boxtimes \mathbb{Z}^{11} \rightarrow \mathbb{Z}^{11}$  on the affine<sup>4</sup> tensor product  $\mathbb{Z}^{11} \boxtimes \mathbb{Z}^{11}$ , defined on the affine basis  $(e_i \boxtimes e_j)$  by  $Q(e_i \boxtimes e_j) = e_{Q(i,j)}$ . For any such address change  $\xi : \mathbb{Z}^{11} \boxtimes \mathbb{Z}^{11} \rightarrow \mathbb{Z}^{11}$ , and any series  $S_P : \mathbb{Z}^{11} \rightarrow P$ , we obtain twelve series in that space by address change  $S_P \cdot \xi$  of the series, and then restricted to the  $i$ th rows of  $\xi$ , or, equivalently, prepending the address change (!)  $row_i : \mathbb{Z}^{11} \rightarrow \mathbb{Z}^{11} \boxtimes \mathbb{Z}^{11}$  defined by  $row_i(e_j) = e_i \boxtimes e_j$ . Using this technique, Boulez constructs such series of series for the two pianos and for the different sections of the compositions by a construction of new address changes  $\xi$ , which uses some simple geometric operations on  $Q$ .

We have implemented these methods on RUBATO® in such a way as to

<sup>3</sup>If  $x$  is an element of a module  $M$ , we denote by  $T^x$  the shifting operation  $y \mapsto x + y$  on  $M$ . If  $M$  is a module of musical pitch(classes),  $T^x$  is called “transposition by  $x$ ”.

<sup>4</sup>This is the universal module with respect to di-affine maps instead of usual linear maps, see [23, E.3.3] for details.

enable generalized compositions, which emerge automatically from the described address change procedure, see [33, 27]. In our composition, entitled *restructures*, that was realized for a presentation to Boulez at the IRCAM in December 2007, we have applied the procedure to twelve instruments; *restructures* can be downloaded from

<http://www.encyclospace.org/special/restructures.mp3>.

### 3.3 Factorization in Global Compositions

The factorization paradigm of musical manifolds, also called global compositions, meets the insight that local structures are not sufficient to describe musical facts: music theory deals with syntactical groupings into periods, melodic units, chord assemblies, voices, or large forms, such as allegro movements. Such groupings' local “charts” are usually neither disjoint nor hierarchically ordered by inclusion. Factorization in this sense means that the overall syntax of musical structures is a composition or layering by local “charts”. Here, the “product” of such charts is their union to build an atlas.

Let us conceive the formal setup of global musical manifolds in order to be able to compare two instances and to calculate their isomorphism classes. It is remarkable that mathematical manifolds were introduced by Bernhard Riemann in his habilitation talk in 1854 [35], the same year where Eduard Hanslick published his celebrated book *Vom Musikalisch Schönen* [12], where he also referred to global musical structures. For a more comprehensive historical account on musical manifolds in theory, analysis, and composition, see [23, ch.13.1]. Recall that one of the most illustrative and meaningful examples of such a global structure is the harmonic band of a 12-tempered diatonic scale  $Dia_x = T^x\{0, 2, 4, 5, 7, 9, 11\}$  on the tonic  $x \in \mathbb{Z}_{12}$ , together with its covering  $Dia_x^{(3)}$  by the seven charts  $I, II, \dots, VII$  of the standard triads. Its nerve  $N(Dia_x^{(3)})$  is a Moebius strip, and the strip's missing orientation is responsible for classical problems in the construction of Riemannian harmony, see [23, ch.13.4.2.1].

Global structures abound in musical performance and in music theory. We therefore give the general definition of a musical manifold. It based on the category of modular local compositions over a commutative ring  $R$ , and at a given address module  $A$ :

**Definition 1** A global modular composition at address  $A$  is given by this data:

- a covering  $I$  of a finite set  $G$  by non-empty subsets  $G_i \subset G, i \in I$ ,
- an atlas  $(K_i)_{i \in I}$  by  $A$ -addressed modular local compositions  $K_i \subset A @ M_i$  in  $R$ -modules  $M_i$ ,
- set bijections  $g_i : G_i \xrightarrow{\sim} K_i$  for all  $i \in I$ ,

- *gluing conditions on the subcompositions  $K_{ij} \subset K_i$  defined by the intersections  $G_i \cap G_j$  and the bijections  $g_i, g_j$ : We have  $R\text{LocMod}$ -isomorphisms  $g_j \circ g_i^{-1}/Id_A : K_{ij} \xrightarrow{\sim} K_{ji}$  for all pairs  $i, j$  of indices.*

With such objects we define the category  $R\text{GlobMod}$  of global  $R$ -modular compositions. The morphisms are the evident maps between the underlying sets, such that they define morphisms on the atlas charts. We omit the details, except the important fact that such a musical manifold is tied to a given covering of the underlying set. This differs from mathematical manifolds, which take the colimit of coverings. Here, this would not make sense: The selection of covering sets is a characteristic feature of a given analytical approach, as we have seen above with the harmonic band. Let us however stress that not every global composition is defined by a covering of an already given local compositions, as was the case with a tonality  $Dia_x^{(3)}$ . There are global compositions, which are not isomorphic to any such covering [23, example 31 in ch.16.1]; they are called *non-interpretable*.

There is a good number of global composition types that can be completely classified. There is even a canonical geometric space, namely a scheme in the sense of Grothendieckian algebraic geometry, such that its ring-valued points correspond to isomorphism classes of certain global modular compositions. We shall discuss this classification theorem for global compositions in the following section.

The more dramatic part of this discussion is the overall nature of objects we have been dealing with to the date. We have seen that local compositions are interpreted as  $A$ -addressed points in powerset presheaves  $\Omega^F$ . This is the topos-theoretic setup to say that these objects are subobjects  $K \subset @A \times F$  defined via characteristic functions  $\chi_K : @A \times F \rightarrow \Omega$ . From the logical interpretation of topos theory, one knows that the set  $A@Omega^F$  of subfunctors of  $@A \times F$  is a Heyting algebra, i.e., an intuitionistic logical algebra defined by the Heyting algebra structure on  $\Omega$  [17, p.201-203]. So local compositions are the logical values in a Heyting algebra, and they are defined as subobjects by Gottlob Frege's characteristic function. This is a dramatic abstraction since nothing must be known here except that a given subobject is just the characteristic morphism fiber over the true point  $T : 1 \rightarrow \Omega$ . This is the maximal abstraction from any content-related properties of the subobject. Nothing needs to be known about the subobject, except that its points have the marker "true". In this language, things are reduced to the *pure facticity* of being true, i.e. in a more Wittgensteinian language, being the case.

This view of topos theory is the Fregean reduction of objects to collections of what is the case, the pure facts. Global compositions are even more so since they are glued from local truth objects, their charts. This is evident for the harmonic band since its nerve is a geometric expression of the intersection configuration, which is the system of logical conjunctions of the degree charts in a diatonic scale that are not "false" (i.e., their intersection is not empty). Although morphisms between local or global compositions enrich the object collection in categories of local compositions, they are not objects, they are external to them. Objects

are strictly factual Fregean ready-mades, we have no access to their generative history, i.e. to the processes that made them.

This logical aspect of the theory of local and global compositions is part of a mathematical attitude, which David Lewin calls “Cartesian” [14, p.20], since it stems from that purely externalist view to things “out there”, namely Descartes’ *res extensae*. They are even less than *res extensae*, since they are only characterized by being the case or not, nothing that would relate to a specific property of the englobing space(-functor) is known; we would call them “Fregean” rather than “Cartesian”. It goes perfectly with this line of thought that Yoneda’s lemma is so central in this topos-theoretic context. In fact, the lemma’s power is that it replaces abstract arrows in categories by natural transformations between set-valued presheaves, which means that we can rephrase abstract arrows in terms of systems of set functions in Frege’s sense. In other words, *the functorial theory of local and global compositions in the framework of the topos Mod<sup>®</sup> is a modern and somehow universal setup of all that can be construed in Frege’s spirit of logically shaped facticity.*

### 3.4 Classification of Global Compositions

Powerset-driven mathematical music theory has been criticized by some music theorists and musicologists for its static and blackbox character since it does not represent any type of musical dynamics, be it in the processual construction of music, be it in its dramatic character that is not only a psychological movement [38, p.169], be it in a more abstract gesture also present in absolute music, such as in Beethoven’s string quartet op. 131, or in Schönberg’s string trio op. 45 [5]. This does not imply that the results from powerset-driven mathematical music theory are irrelevant, it merely asks for an unwrapping of these hermetic formulas.

#### 3.4.1 Networks

In his attempt to open the Fregean prison, Lewin has proposed a “transformational theory” [14, p.159], which is oriented towards the famous question: “If I am at  $s$  and wish to get to  $t$ , what characteristic gesture should I perform in order to arrive there?” His proposal is however not the full-fledged solution of that question since he in fact introduced movements, but no real gestures. His theory converged to a processual language, which is in a sense a rediscovery of category theory in music, and which has become quite prominent under the title of *Klumpenhouwer networks* [15]. Let us see how such networks open the Fregean prison.

We first discuss this for sets. Consider a subset  $K \subset X$  of a set  $X$ , the set-theoretic characteristic function  $\chi_K : X \rightarrow \Omega$ , maps the elements of  $K$  to  $1 \in \Omega$  this means that we have a commutative diagram of sets

$$\begin{array}{ccc} 1 & \xrightarrow{k} & X \\ !\downarrow & & \downarrow \chi_K \\ 1 & \xrightarrow{\top} & \Omega \end{array}$$

In other words, the elements  $k \in K$  are the solutions  $x = k$  of the “Fregean equation”  $\chi_K \circ x = \top$ . We symbolize this by the arrow representation  $\chi_K/! : x \rightarrow \top$ , which relates the “variable”  $x$  to the “fixed truth element”  $\top$  by the “Fregean arrow”  $\chi_K/!$ . In this rephrasing of set-theoretic elements, the network idea is a straightforward generalization. One first generalizes the fixed truth element to a variable and then also generalizes the characteristic function to more general functions, and third, one generalizes the simple setting for one function to a diagram of such functions. More precisely, for any category  $\mathcal{C}$  we take the category  $\int_{\mathcal{C}}$  of  $\mathcal{C}$ -points. Identifying objects  $A \in \mathcal{C}$  with the representable functors  $@A \in \mathcal{C}^{\otimes}$  via Yoneda, the objects of  $\int_{\mathcal{C}}$  are the natural transformations  $x : A \rightarrow F$ , where  $A \in \mathcal{C}$  and  $F \in \mathcal{C}^{\otimes}$ , (i.e., by Yoneda, the elements  $x \in F(A)$ ). The morphisms from  $x : A \rightarrow F$  to  $y : B \rightarrow G$  are the pairs  $f/\alpha$  such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{x} & F \\ \alpha \downarrow & & \downarrow f \\ B & \xrightarrow{y} & G \end{array}$$

This is the evident generalization of the above Fregean diagram. Then a  $\mathcal{C}$ -network is just a diagram  $\delta : \Delta \rightarrow \int_{\mathcal{C}}$  of  $\mathcal{C}$ -points. Let us give two musically relevant examples of such a network to illustrate the power of this conceptualization, see [26] for an in-depth discussion. The first example is prototypical for the change of paradigm propagated by Lewin. In [18, p.30-32], we considered chords (sets of pitch classes in  $\mathbb{Z}_{12}$ ) that were defined by networks of simple diagrams. One prominent example is the major chord  $Tc = \{0, 4, 7\}$ . As a set it has no internal structure, no process that tells us how this chord has been generated, it is just “out there”. This deficiency is solved by a  $\mathbb{Z}\text{Mod}$ -network

$$\delta = 0 \xrightarrow{T^7 3} 7 \xrightleftharpoons[T^7 3]{T^7 3} 4$$

over the digraph<sup>5</sup>  $\Delta = \bullet \longrightarrow \bullet \xleftarrow{\quad} \bullet$ , at the zero-address, with three representable spaces  $@\mathbb{Z}_{12}$  in its vertices, three identical space morphisms  $T^7 3$  and the zero address change everywhere. The numbers  $x = 0, 7, 4$  in the vertices stand for the points  $x : 0 \rightarrow @\mathbb{Z}_{12} : 0 \mapsto 0, 7, 4$ . We have classified all chords generated by one single map from one selected point (like here with the map  $T^7 3$  and the point 0) in [18, p.30-32]. These circle chords represent most of

<sup>5</sup>We choose the terminology “digraph” from category and graph theory for a directed multigraph here; in representation theory, they call it a “quiver”.

the standard chord in Western harmony and they are at the origin of Noll's reconstruction of Riemannian harmony [34].

Networks are still another type of factorization. In this case it is not a decomposition of an object as a product of objects of the same type (like group elements written as products of generators), but the action of a diagram of function arrows on a collection of objects, generating their orbits under the products of the diagram's arrows. So an amorphous set (of objects sitting in the network's vertices) is deployed as a "narrative" of objects that are transported by the diagram's chains of transformations. The essential here is that the factorization now penetrates the set's anatomy, it is an *internal* factorization, not a juxtaposition of external factors.

Zero-addressed networks with spaces  $\mathbb{Z}_{12}$  in their vertices, and affine isomorphisms in their arrows, are the already mentioned Klumpenhouwer networks and are the core objects of transformational theory. The necessity for more general addresses is easily shown by the following example of a network of dodecaphonic series  $s$ , which are points  $\mathbb{Z}^{11} \rightarrow \mathbb{Z}_{12}$ . In the following network, the symbols  $Us$ ,  $Us$ ,  $UKs$  stand for inversion, retrograde, retrograde inversion of  $s$ , respectively:

$$\begin{array}{ccc} s & \xrightarrow{Id/T^{11}.-1} & Ks \\ T^{11}.-1/Id \downarrow & & \downarrow T^{11}.-1/Id \\ Us & \xrightarrow{Id/T^{11}.-1} & KUs \end{array}$$

The retrograde  $Id/T^{11}. - 1$  is a pure change of address without space transformation, as already discussed earlier; the inversion  $T^{11}. - 1/Id$  is a pure space transformation without address change.

From the mathematical point of view, networks relate to the second big construction defining a topos: limits (and their duals, colimits). Intuitively, limits are the generalization of a system of equations. This is exactly, what we have considered when defining the category  $\int_{\mathcal{C}}$  of  $\mathcal{C}$ -points. With the above notations, a morphisms  $f/\alpha : x \rightarrow y$  in this category is a pair of points  $x, y$  that solves the equation  $f \circ x = y.\alpha$  defined by a diagram of two arrows:  $A@F \xrightarrow{A@f} A@G \xleftarrow{\alpha@G} B@G$ . Therefore, a  $\mathcal{C}$ -point  $\delta : \Delta \rightarrow \int_{\mathcal{C}}$  is an element of the limit of the diagram derived from  $\delta$  by inserting the two opposite "numerator" / "denominator" arrows for each arrow of  $\Delta$ .

### 3.4.2 The Classification Network of Global Compositions

So far, we have shown that the passage from powerset constructions to network processes is a natural extension of the factual Fregean truth equation  $\chi_K \circ x = \top$  to a system of more general equations which are represented by diagrams of  $\mathcal{C}$ -points. We have also seen that the solutions of such diagrams are essentially limit elements, and that this completes the topos-theoretic toolbox. So the passage is that from a static declaration of elements to a system of equations, which, as a kind of industrial plant, evaluates possible candidates (the  $\mathcal{C}$ -points in their vertices) for solutions of the given equations. The resulting networks

are not only sets, but their elements are connected by generating equational conditions. There might be no solutions, and the defining equations are still given. This shifts the focus from the solutions (the factual objects) to the construction scheme (the “industrial” processes).

Beyond this conceptual shift, there is also a beautiful theoretical shift to networks that occurs while elaborating on the theorem of classification of global compositions. This means that even without thinking of switching to networks, global compositions enforce networks, when we embark in the calculation of their isomorphism classes. Fortunately, we can expose the core ideas without delving in those very technical details of the algebraic geometry of moduli spaces. The classification of global compositions makes use of a classical device, namely desingularization. In our situation this means that in a global composition, its points are usually not in general positions. For example, it might happen that three points in the onset-pitch space are collinear instead of defining a triangle; or that four points in the onset-pitch-loudness space are coplanar, or even collinear instead of defining a tetrahedron. This is a major reason, why there are global compositions that are not interpretable (we have mentioned this fact in subsection 3.3). The desingularization of a global composition  $G^I$  therefore is the first step towards a better understanding of global compositions. Every  $G^I$  has such a desingularization, a global composition  $\Gamma$  which is called *resolution* of  $G^I$ . Combinatorially speaking it is not different from  $G^I$ : It has the same number of points and the same size and configuration of charts. But within each chart  $\Gamma_i$  of  $\Gamma$ , the points are in general position, which means that if  $\Gamma_i$  has  $k_i$  points, then their pairwise differences in the ambient space generate a space of dimension  $k_i - 1$ . For example, if  $k_i = 4$ , then the three differences generate a three-dimensional space, or else: the four points define a tetrahedron. The example in figure 1 shows such a global composition. It has six points 1,2,...,6, three charts:  $G_1 = \{1, 2, 3, 4\}$ ,  $G_2 = \{1, 2, 5, 6\}$ ,  $G_3 = \{3, 4, 5, 6\}$ , which are all singular as their four points lie in a plane. The charts are glued on three pairs of points: (1,2), (5,6), and (3,4). The pathology here is the gluing of points 1,2, which is twisted, as shown in the figure. This global composition is not interpretable. But its resolution  $\Gamma$  is interpretable, as shown in the figure: its charts lie in a three-dimensional space and are non-singular. In fact, each of the three charts  $\Gamma_1, \Gamma_2, \Gamma_3$  generates a tetrahedron.

The classification now resides on two facts: First, the resolution is a “free” object, it has no hidden parameters. Second, the affine functions on  $G^I$  can be used to reconstruct  $G^I$  from its resolution as follows: We take the nerve  $N(\Gamma)$ , which is mapped bijectively onto the nerve  $N(G^I)$  under the bijective *res* morphism (which is not an isomorphism, in general!) as shown in figure 1. For every simplex  $\sigma \in N(\Gamma)$  the module  $\mathbf{Aff}(\text{res}(\sigma))$  of affine functions on the intersection of the charts in the image simplex  $\text{res}(\sigma)$  can be retracted by *res* to a submodule  $F_\sigma \subset \mathbf{Aff}(\sigma)$  of the module of affine functions on the intersection of the charts in the simplex  $\sigma$ . Taking for every point  $x$  in a simplex  $\sigma$  of  $\Gamma$  its evaluation on  $F_\sigma$  yields an embedding  $x \mapsto \dot{x} \in F_\sigma^*$  in the linear dual module of  $F_\sigma$ . It can be shown that this rebuilds  $G^I$  up to isomorphism for good conditions as stated in the classification the following classification theorem [23, Theorem

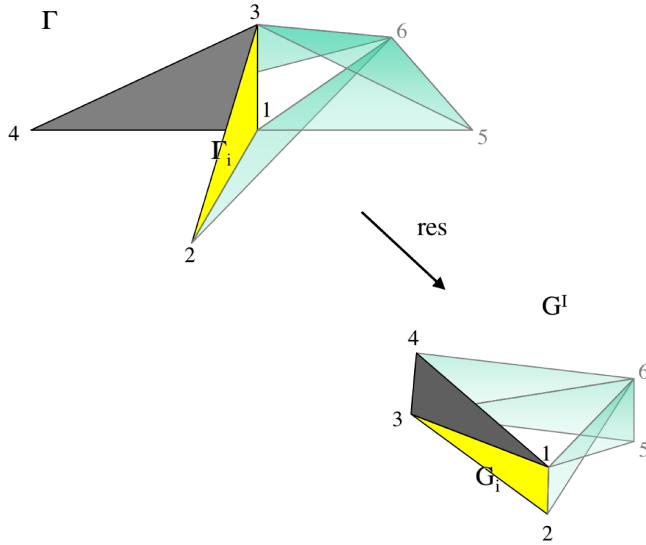


Figure 1: The resolution  $\Gamma$  of a global composition  $G^I$ . It is mapped one-to-one onto  $G^I$ , and chart  $\Gamma_i$  corresponds to chart  $G_i$ .

18, ch.15.3.2]:

**Theorem 1** (Geometric Classification of Global Compositions)

Let  $A$  be a locally free module of finite rank over the commutative ring  $R$ . Consider the  $A$ -addressed global compositions  $G^I$  with the following properties (\*):

- the modules  $R.G_i$  that are generated by the charts  $G_i$ , i.e., by all differences  $x - y, x, y \in G_i$ , within the chart spaces, are locally free of finite rank;
- the modules of affine functions  $\mathbf{Aff}(G_i)$  are projective.

Then there exists a subscheme  $J$  of a projective  $R$ -scheme of finite type whose points  $\omega : \text{Spec}(S) \rightarrow J$  are in one-to-one correspondence with the isomorphism classes of global modular compositions at address  $S \otimes_R A$  sharing the properties (\*).

The reconstruction of the global composition  $G^I$  from affine functions is based upon the total system of such functions on the nerve of the resolution  $\Gamma$ . More precisely, the nerve  $N(\Gamma)$  defines a digraph whose vertices are the simplexes  $\sigma$ , and whose arrows  $\sigma \rightarrow \tau$  are the inclusions  $\sigma \subset \tau$ . We have the corresponding linear restriction maps  $r(\sigma, \tau) : \mathbf{Aff}(\sigma) \rightarrow \mathbf{Aff}(\tau)$  for each couple  $\sigma \rightarrow \tau$ . If we take the space  $S(\mathbf{Aff}(\sigma))$  of linear subspaces of  $\mathbf{Aff}(\sigma)$  for each simplex  $\sigma$ , then the restriction  $r(\sigma, \tau)$  induces a map  $S(\sigma, \tau) : S(\mathbf{Aff}(\sigma)) \rightarrow S(\mathbf{Aff}(\tau))$  by projection. This defines a diagram of sets of subspaces, and it is clear that the subspaces  $F_\sigma \in S(\mathbf{Aff}(\sigma))$  stemming from  $G^I$  as described above

form an element of the limit  $A @ \lim_{N(\Gamma)} S(\mathbf{Aff}(\sigma))$  (we write this as the value of a presheaf at address  $A$ , since this address is implicitly chosen and in fact defines a presheaf, see [23, ch.15,16] for details). In other words: *The retracted affine functions from  $G^I$  are ( $A$ -addressed) networks that can be classified on the free global composition defined by the resolution  $\Gamma$  of  $G^I$ .*

Summarizing, we have shown that the limit part of the topos-theoretic perspective in music theory is musically as well as theoretically required to enrich our conceptual power towards processes and our analytical understanding the classification of powerset objects. In other words, *the factorization via networks is necessary to fully understand the facticity of powersets*. This is a beautiful dialectic movement in Hegel's sense, where a concept (the facticity of powersets) transcends its conceptual "skin" and generates the negation of its negation, what Hegel calls affirmation, namely the network concept.

To complete this image, we should add that certain types of local and global<sup>6</sup> networks allow for a construction of functorially associated (local or global) compositions, and thereby, the classification of the latter can help provide a relative classification of networks, see [25] for details. The absolute classification of networks is however still far from settled.

## 4 Factorization with Gestures

So far, the mathematical apparatus of topos theory has been useful to establish the Fregean level of powerset-driven facticity in music theory, it has also been an essential conceptualization for the processual unfolding of facticity by use of limits, and we have learned that even the classification of powerset objects (global compositions) can be modeled upon the limit-driven part of topos theory. So, what is missing? Isn't this enough to admit that topos theory is the key to all our needs in the enterprise of understanding and managing music?

It is not, of course, since in the making of music, in its performance, those processes described by diagrams of  $\mathcal{C}$ -points lack of the embodiment in gestures, be they real physical movements, or more abstract entities in spaces of musical thought and drama. To understand the problem, let us come back to Lewin's famous sentence: "If I am at  $s$  and wish to get to  $t$ , what characteristic gesture should I perform in order to arrive there?" He later adds that he thinks the subject "T" of this question not as an abstract voyager and observer, but as a dancer who moves within music, much as Helga de la Motte's idea [7] of thinking about music as being a virtual identification with the musical object.

The reference to gestures, to embodiment and to a theory of these phenomena in music is quite prominent. Let us cite once again Adorno [2, p.247]: "Danach wäre die Aufgabe des Interpreten, Noten so zu betrachten, bis sie dem insistenten Blick im Originalmanuskripte sich verwandeln; nicht aber als Bilder der Seelenregung des Autors — sie sind auch dies, aber nur akzidentiell — sondern als die seismographischen Kurven, die der Körper der Musik selber in seinen

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<sup>6</sup>Our previous construction of networks is in fact one of *local* networks, and global networks can easily be defined by gluing local networks using standard procedures [25].

gestischen Erschütterungen hinterlassen hat.” The interpreting performer has to penetrate the score until the gestural vibrations come back to life and reveal the music’s body. This reference to embodiment is also shared by Cecil Taylor, the *monstre sacré* of free jazz piano: “The body is in no way supposed to get involved in Western music. [...] I try to imitate on the piano the leaps in space a dancer makes.” This statement is all the more relevant as it opens the definition of music to non-score-related creation. Music might be mediated through scores, but both, Adorno’s and Taylor’s statements, reveal that the deeper level of creativity is beyond the notes, more in a kind of body and its gestural utterances.

Interestingly, mathematics is not prepared for such a conceptualization. This is evident in Gilles Châtelet’s statement in [4]: The Fregean function’s externalism is opposed to the gesture’s intensity. With respect to gestures, the great mathematician Henri Poincaré was very sensitive in his understanding of what is a space, and what are the points in a space (cited after [16]): “Localiser un objet en un point quelconque signifie se représenter le mouvement (c’est-à-dire les sensations musculaires qui l’accompagnent et qui n’ont aucun caractère géométrique) qu’il faut faire pour l’atteindre.” This fits in the etymology of space, namely *ex pati*, to reach out, to walk through. A space is what you walk through. It is defined by your walking gesture. Modern mathematics has forgotten about that, however conserving the intuitive roots in its descriptive language. Linear algebra, the prototypical theory of our times, reduces spatial movements, such as a spatial rotation around an axis in  $\mathbb{R}^3$ , to an abstract formula, a  $3 \times 3$ -matrix in our example, whose suggested motion is completely absent in the formula. It takes some time to extract this rotational movement from the matrix: Search for eigenvectors, and then for the angle, etc.

#### 4.1 The Missing Part in Yoneda’s Lemma

It might be questionable if one seeks to represent mathematics by sensual categories. After all, the attempt to reduce mathematics to cognitive categories of the human psyche fails as it is built on a vicious circle: Any such reduction would need mathematics to prove its validity. We are not trying to *reduce* mathematics, but to bring its manipulation back to sensual human activities. Let us make this clear with Yoneda’s lemma: Abstract categories have packed structures so densely that their manipulation has become the famous “abstract nonsense”: Juggling arrows is bare of any imaginable sense, the only intuition one is left with is the arrow graphics that might help manipulate some of the elementary operations with categories. In this context, Yoneda’s lemma gives us back the Fregean intuition of arrows, namely functions  $g$  that transform arguments  $x$  into values  $g(x)$ . The natural transformation  $@f : @X \rightarrow @Y$  associated with an abstract arrow  $f : X \rightarrow Y$  in a category  $\mathcal{C}$  represents that arrow as a system of Fregean functions on sets  $A@f : A@X \rightarrow A@Y$  for each address  $A$ . And the lemma rightly tells us that this representation is faithful: Whatever you do with those systems generates an arrow in the given category, you are allowed to act in a Fregean sense and you will generate some abstract

arrow, say.

This is what we mean when asking for sensual representation. In our context, it is the representation of mathematical activity by gestures, which is being thematized. This is exactly what is missing in Yoneda's lemma: We have transferred abstract arrows to (systems of) Fregean functions  $g$ . But these functions are still defective in the sense of Châtelet: they are teleportations, nothing is known about the path from  $x$  to  $y = g(x)$ , that path is only imagined, or, to put it more honestly: it is a fiction. Set theory has rightly reduced the function concept to its brute graph: there is no more connectivity between argument  $x$  and function value  $y$ , than just combining them to an ordered pair  $(x, y)$ . Why can't we dream of giving this teleportation back the original idea of *moving*  $x$  to  $y$ ? It must be possible to complete Yoneda's lemma to a theorem, where abstract arrows are really restated as gestures spatially moving  $x$  to  $y$ . The present lemma is only what can be called the "objective" part (the category's objects have been transformed in systems of sets), the gestural, "morphic" part is still missing.

It is remarkable, but not unexpected, that this mathematical topic is raised by music theorists. The reason might be that forms in music, such as the score notation, are always known to be crutches to the intended expression, whereas in mathematics, the formal apparatus is so dominant that one easily forgets about the expressive background. Lewin's famous sentence is typical for this phenomenon in music: He means gestures, but his formalism is straightforward set-theoretic category theory. Musicians know that their notations are way behind their intentions and therefore never wonder about formal deficiencies. Or, stated more provocatively in terms of the opposite movements between formulas and gestures: *Musicians unpack formulas to gestures and do not care too much about the wrapping, whereas mathematicians wrap gestures to formulas, and their main concern is the wrapping, not its gestural content.*

Nevertheless, there are some attempts to resuscitate those "frozen gestures" in mathematics. Let us mention two of them: The first is Gabriel's theory of quiver representations [10]. He provides algebra with a graph-theoretic flavor. For example, the algebra  $\mathbb{C}\langle X, Y \rangle$  of non-commutative polynomials in two variables  $X, Y$  and complex coefficients is represented by the complex linear combinations of paths built upon the two loops  $X, Y$  of digraph  $\Delta = \begin{array}{c} \text{Y} \\ \text{---} \\ \text{X} \end{array}$ . More generally, every directed graph  $\Gamma$  gives rise to a quiver algebra  $R\Gamma$  over a commutative ring  $R$ . Our example is  $\mathbb{C}\langle X, Y \rangle \xrightarrow{\sim} \mathbb{C}\Delta$ . This theory is half way down from abstract algebra to gesture theory, and it is in fact the theory needed for Lewin's transformational theory.

The second example is more substantial. One might wonder about the essential step that is taken when moving from the field  $\mathbb{R}$  of real numbers to the algebraically closed field  $\mathbb{C}$  of complex numbers. The decisive difference is, of course, the fundamental theorem of algebra: every non-constant polynomial with complex coefficients has a root in  $\mathbb{C}$ . This seemingly algebraic fact is however based upon properties of the complex plane that are thoroughly topological. It has a beautiful proof that is using homotopy theory, see Lages [13, ch.5.1]. And this is a strong move from algebra to algebraic topology, to the

theory of continuous curves and their homotopic deformations. We shall come back to this theory later. For the moment being, we want to bring to the point the topological difference between  $\mathbb{R}$  and  $\mathbb{C}$ . This is manifest when mapping  $x$  to  $-x$  in  $\mathbb{R}$ . This mirror function cannot be thought as a continuous movement, since any continuous curve from  $x$  to  $-x$  would hit the origin 0 of  $\mathbb{R}$  and become singular there. The mirroring operation is a mysterious thing in the reals, it is not a movement, but a real teleportation in Frege's sense. However, in  $\mathbb{C}$ , negation is a simple rotation. We may reach  $-x$  from  $x$  on a continuous circular curve  $c(t) = xe^{i\pi t}$ ,  $t \in I = [0, 1]$ . And the pivotal complex number, the imaginary unit  $i = \sqrt{-1}$ , is exactly in the middle of this rotation:  $i = e^{i\pi/2}$ , where the real part vanishes.

The above discussion suggests that one may envisage a mathematical theory of gestures, which would at the end enable the completion of Yoneda's lemma in its morphic part and, as required by music theorists and musicians, also offer a more explicit description and analysis of gestural layers in music. We have developed such a theory in [29] for gestures in topological spaces, and in [30] for gestures in topological categories. First applications of the theory to the understanding of flow and collaboration in free jazz were described in [31], an application to gestures for tonal modulations in Beethoven's op. 106 are described in [30].

## 4.2 Topological Gesture Theory

Despite the intuitive understanding of what is a gesture, including body movement and semantics, a precise conceptualization looks less easy. We agree with Jean-Claude Schmitt [36] that the medieval definition of a gesture, as given by Hugues de Saint-Victor, remains one of the most adequate, at least when referring to the concrete human body: “Gestus est motus et figuratio membrorum corporis, ad omnem agendi et habendi modum.” Gesture is the movement and figuration of the body’s limbs with an aim, but also according to the measure and modality proper to the achievement of all action and attitude. In the following definition of a gesture, we shall rely on Saint-Victor’s definition and implement the (con)figuration of a gesture by the articulation of diagrams. We shall then describe the movement in the parametrization of curves representing the figuration, and we shall formalize the body’s space-time by a topological space, where the movement takes place. The semantics of gestures will not be our concern here, this must be dealt with after a thorough investigation of the formal mathematics of gestures.

Let us first review the topos *Digraph* of digraphs. Its objects are functions  $\Gamma : A \rightarrow V^2$  from a set  $A = A_\Gamma$  of *arrows* to the Cartesian square  $V^2 = V \times V$  of the set  $V = V_\Gamma$  of *vertexes*. The first projection  $t = pr_1 \circ \Gamma$  is called the *tail* function, the second  $h = pr_2 \circ \Gamma$  is called the *head* function of the digraph. For an arrow  $a$ , the vertexes  $t(a)$ ,  $h(a)$  are called its head and tail, respectively, and denoted by  $t(a) \xrightarrow{a} h(a)$ . A morphism  $f : \Gamma \rightarrow \Delta$  of digraphs is a pair  $f = (u, v)$  of functions  $u : A_\Gamma \rightarrow A_\Delta$ ,  $v : V_\Gamma \rightarrow V_\Delta$  such that  $v^2 \circ \Gamma = \Delta \circ u$ .

We now need a special subcategory of digraphs, the *spatial digraphs*. Such a digraph is associated with a topological space  $X$  and denoted by  $\vec{X}$ . By definition, the arrow set is  $A_{\vec{X}} = I@X$ , the set of continuous curves  $c : I \rightarrow X$  in  $X$ , while the vertex set is  $V_{\vec{X}} = X$ ,  $h(c) = c(1)$ , and  $t(c) = c(0)$ . A spatial morphism is a digraph morphism  $\vec{f} : \vec{X} \rightarrow \vec{Y}$  canonically induced by a continuous map  $f : X \rightarrow Y$ . The subcategory of spatial digraphs and morphisms is denoted by *SpaceDigraph*. A spatial digraph is more than a digraph: it is also a topological digraph in the following sense. The set  $A_{\vec{X}} = I@X$  of arrows of  $\vec{X}$  is a topological space by the compact-open topology, and the head and tail maps  $h, t : I@X \rightarrow X$  are continuous. Moreover, for a continuous map  $f : X \rightarrow Y$ , the arrow map  $I@f : I@X \rightarrow I@Y$  is continuous.

**Definition 2** *Given a digraph  $\Delta$  and a topological space  $X$ , a (topological)  $\Delta$ -gesture in  $X$  is a digraph morphism  $\delta : \Delta \rightarrow \vec{X}$ , i.e., a realization of the abstract vertexes and arrows within a topological space. Given two gestures  $\delta : \Delta \rightarrow \vec{X}, \gamma : \Gamma \rightarrow \vec{Y}$ , a morphism  $f : \delta \rightarrow \gamma$  is a digraph morphism  $f : \Delta \rightarrow \Gamma$  such that there exists a spatial morphism  $\vec{h} : \vec{X} \rightarrow \vec{Y}$  which commutes with  $f$ , i.e.,  $\vec{h} \circ \delta = \gamma \circ f$ . This defines the category *Gesture* of topological gestures.*

The important property of this category is that it enables iterated gestural constructions, so-called hypergestures: gestures of gestures. The reason is that the set  $\Delta @ \vec{X}$  of  $\Delta$ -gestures in  $X$  is a topological space in a natural way, induced by a limit construction from the compact-open topology on the space  $I@X$ ; it is denoted by  $\Delta \vec{\otimes} X$ . We may therefore consider spaces of hypergestures  $\Gamma \vec{\otimes} \Delta \vec{\otimes} X$ , of hyperhypergestures  $\Sigma \vec{\otimes} \Gamma \vec{\otimes} \Delta \vec{\otimes} X$ , etc. Observe that a hypergesture is a natural generalization of curve homotopy, therefore hypergestures are natural objects referring to algebraic topology. Hypergestures have intuitively been considered by a number of musicians and theorists, among them Adorno's student Renate Wieland, a famous piano pedagogue [38, p.190]: "Die Klangberührung ist das Ziel der zusammenfassenden Geste, der Anschlag ist sozusagen die Geste in der Geste." Higher hypergestures are connected to each other by what I call the Escher theorem, since the permutations of digraphs relate to Maurits Cornelis Escher's ambiguous works of art:

**Theorem 2** (Escher Theorem) *If  $\Gamma, \Delta$  are digraphs and  $X$  is a topological space, then we have a canonical homeomorphism  $\Gamma \vec{\otimes} \Delta \vec{\otimes} X \xrightarrow{\sim} \Delta \vec{\otimes} \Gamma \vec{\otimes} X$ .*

This means that one may change the perspective of the "outer" digraph and thereby look at hypergestures from variable digraph perspectives. This is particularly useful for creative processes in jazz improvisation [31, ch.9].

### 4.3 Gestoids

The generalization of homotopy by hypergestures refers to the well-known category  $H_X$  of homotopy classes of curves in  $X$ . More precisely, its objects are

the elements of  $X$ , while the morphism set  $H_X(x, y)$  is the set of homotopy classes of curves starting in  $x$  and terminating in  $y$ . The composition of homotopy classes is the homotopy class of the composed curves. Clearly, this is a groupoid, the inverse of a curve class  $[\gamma]$  being the class  $[\gamma^*]$  of the inverted curve  $\gamma^*(t) = \gamma(1 - t)$ . In particular, the group  $H_X(x, x)$  is the fundamental group  $\pi_1(x, X)$  of  $X$  in  $x$ . The category  $H_X$  is called the fundamental groupoid of  $X$ . If  $\delta : \Delta \rightarrow \vec{X}$  is a gesture, the groupoid generated by the arrows and point of  $\delta$  via the canonical morphism  $\Delta \xrightarrow{\delta} \vec{X} \rightarrow \text{Path}(\vec{X}) \rightarrow H_X$  is denoted by  $H_\delta$  and called *the fundamental groupoid of  $\delta$* . For example, if  $\Delta = 1$  is the final digraph (one vertex, one loop), and if  $\delta : 1 \rightarrow \vec{X}$  is a loop in  $x$ , then  $H_\delta$  is the subgroup of the fundamental group  $\pi_1(x, X)$  generated by the homotopy class of  $\delta(T)$ . For example, if  $X = S^1 \subset \mathbb{C}$ , the unit circle, and if  $\delta$  maps the loop to a one-time rotation around the origin  $1 \in S^1$ , then  $H_\delta \xrightarrow{\sim} \mathbb{Z}$ , the fundamental group of  $S^1$ .

We now linearize the fundamental groupoid of  $\delta$  over  $\mathbb{C}$ , i.e., the sets  $H_\delta(x, y)$  are taken as a basis over the complex numbers, and the composition is defined by bilinear extension from the given basis composition. We call this category  $\mathbb{C}\text{-}H_\delta$  the *gestoid of  $\delta$* . In the above example of the unit circle, we have  $\mathbb{C}\text{-}H_\delta \xrightarrow{\sim} \mathbb{C}\mathbb{Z}$ , the group algebra of  $\mathbb{Z}$  over the complex numbers. Observe that this is essentially the algebra of finite Fourier polynomials if we take as generator the curve  $t \mapsto e^{2\pi it}$ . In this interpretation, these Fourier expressions are linear combinations of homotopy classes of gestures (i.e., particular hypergestures). This perspective opens a gestural understanding of Fourier's formula, see [31, ch.11.1] for details.

The (hyper)gestural interpretation of the fundamental group is a basic tool in the gestural reconstruction of abstract group theory since every group is isomorphic to the fundamental group of some topological space, see Spanier [37, p.147]. This calls for a program of rethinking group theory in terms of hypergestures and to allow access to this central field of mathematics (recall the classification of finite simple groups!) on the basis of gestural representation and manipulation.

#### 4.4 Categorical Gesture Theory

At this point, the mathematically hardest part of gesture theory: a morphic completion of Yoneda's lemma, is still unsettled. We want to discuss this last topic and to propose a path to its solution. But let us first summarize the essential problem. We have general categories to deal with. Their morphisms are abstract arrows, and we would like to "materialize" them in terms of gestures, much as abstract objects in categories have been "materialized" in Yoneda's lemma in terms of sets via representable set-valued functors.

The idea is to reinterpret the concept of a topological gesture in a general categorical context such that it can be applied without the special topological environment. In order to conceive of a general method for generating gestures from morphisms  $f : X \rightarrow Y$  in abstract categories, we start with a heuristic

consideration. Let us work in a musical parameter space  $\mathbb{R}^2$ , parametrizing onset and pitch, for example. We endow it with the structure of the Gaussian plane of complex numbers. Look at a rotation  $e^{i\theta} : x \mapsto xe^{i\theta}$  on  $\mathbb{R}^2$ . Although this morphism  $f = e^{i\theta}$  acts by Fregean “teleportation” on  $x$ , our intuition of a rotation by angle  $\theta$  is different in that we imagine a continuous rotational movement of  $x$  around the space origin until it reaches the final position  $xe^{i\theta}$ . This process is visualized by the trace of  $x$  while rotating, i.e., by a continuous curve  $c_x : I \rightarrow \mathbb{R}^2 : t \mapsto xe^{i\theta t}$  on a circle of radius  $|x|$ . On this curve, each intermediate position  $xe^{i\theta t}$  corresponds to a factorization  $f = e^{i\theta(1-t)} \circ e^{i\theta t} = f_{1-t} \circ f_t$  of  $f$ . In other words, the curve  $c : I \rightarrow \text{GL}(2, \mathbb{R})$  is a curve of factorizations of the given morphism  $f$ . This restatement of the gesture  $c$  in terms of factorizations means that  $c$  is viewed as being an “infinite” factorization insofar as the factors are parametrized by the curve parameter  $t \in I$ . A gesture is therefore rethought as being related to an infinite factorization of the given arrow into a sequence of “infinitesimal” arrows.

This enables us to rethink the basic elements of a gestural interpretation of morphisms in abstract categories. To this end, we fix a morphism  $f : X \rightarrow Y$  in a category  $\mathcal{C}$ . The *category*  $[f]$  of factorizations of  $f$  is defined as follows. Its morphisms are the triples  $(u, g, v)$  of morphism  $u : X \rightarrow W, g : W \rightarrow Z, v : Z \rightarrow Y$  such that  $v \circ g \circ u = f$ . The domain map is  $d(u, g, v) = (u, \text{Id}_W, v \circ g)$ , while the codomain map is  $c(u, g, v) = (g \circ u, \text{Id}_Z, v)$ . Suppose we have two morphisms  $(u, g, v), (r, h, s)$  such that  $c(u, g, v) = d(r, h, s)$ ,  $h : Z \rightarrow Q$ , then their composition is the morphism  $(u, h \circ g, s)$ , as shown in the following commutative diagram:

$$\begin{array}{ccccc}
 & & X & & \\
 & \swarrow u & \downarrow r & \searrow & \\
 W & \xrightarrow{g} & Z & \xrightarrow{h} & Q \\
 & \searrow v & \downarrow & \swarrow s & \\
 & & Y & &
 \end{array}$$

If  $\mathcal{C}$  is a topological category<sup>7</sup>, then so is  $[f]$ , if it is viewed as a subset of  $\mathcal{C}^3$ . For any two objects  $X, Y$  in  $\mathcal{C}$  we now build the disjoint sum  $[X, Y] = \coprod_{f \in X @ Y} [f]$  of the factorization categories  $[f]$  (including the coproduct of topologies on the  $[f]$ ).

Here are two basic examples of topological categories: (1) The *simplex category*  $\nabla$  associated with the unit interval  $I$ . This is the category that will replace the topological space  $I$  in our categorical approach to gestures. Its morphism set is  $\nabla = \{(x, y) | x, y \in I \text{ and } x \leq y\}$ ,  $d(x, y) = (x, x)$ ,  $c(x, y) = (y, y)$ , the composition of morphisms is obvious, and the topology on  $\nabla$  is the relative topology inherited from the usual product topology on  $I \times I \subset \mathbb{R} \times \mathbb{R}$ . (2) The *graph category* associated with any topological space  $X$ . This one guarantees that the

<sup>7</sup>This means that  $\mathcal{C}$ —viewed as being the set of arrows—is a topological space such that the composition and the identity maps are continuous.

topological gesture theory can be embedded in the categorical one. This category's morphism set is  $X \times X$ , equipped with the product topology, while we set  $d(x, y) = (x, x)$ ,  $c(x, y) = (y, y)$ , and again, the composition of morphisms is the obvious one. If no confusion is likely, we denote the graph category of  $X$  by  $X$ . Clearly, a graph category is a topological groupoid. In particular, the simplex category  $\nabla$  is just the subcategory of the graph category  $I$  on the pairs  $(x, y), x \leq y$ .

The next step is to redefine curves in such general topological categories. In the topological setup, a curve was a continuous map  $c : I \rightarrow X$  into a topological space  $X$ . Here, we have to consider topological categories  $\mathcal{C}$  instead and then also maps that preserve the factorization in such categories. This is achieved by defining curves as continuous functors  $c : \nabla \rightarrow \mathcal{C}$  with values in a topological category  $\mathcal{C}$ . In the case of the graph category of a topological space, this concept coincides with the topological one.

Therefore, when considering curves in factorization categories, we have

$$\nabla @ [X, Y] = \coprod_{f \in X @ Y} \nabla @ [f],$$

and, if we endow  $[X, Y]$  with the coproduct topology, also the set of continuous functors (symbolized by  $\circledcirc$ ) is  $\nabla \circledcirc [X, Y] = \coprod_{f \in X @ Y} \nabla \circledcirc [f]$ .

In order to obtain gestures in topological categories, we need to mimic the construction of a spatial digraph defined in section 4.2. To this end, we consider the two continuous tail and head functors  $t, h : \nabla @ \mathcal{C} \rightarrow \mathcal{C}$ , which are defined as follows. If  $\nu : f \rightarrow g$  is a natural transformation between  $f, g : \nabla \rightarrow \mathcal{C}$ , then  $t(\nu) = \nu(0) : f(0) \rightarrow g(0)$ , and  $h(\nu) = \nu(1) : f(1) \rightarrow g(1)$ . So the tail and head maps are not only set maps but functors. Call this diagram of topological categories and continuous functors the *categorical digraph*  $\vec{\mathcal{C}}$  of  $\mathcal{C}$ . If we forget about the category and just retain the objects of this configuration, we call it the *(underlying) spatial digraph* of  $\mathcal{C}$ . In particular, if  $\Gamma$  is a digraph, the set of morphisms  $\Gamma @ \vec{\mathcal{C}}$  is the set of digraph morphisms into the underlying spatial digraph of  $\mathcal{C}$ . In other words, such a morphism assigns an object of  $\mathcal{C}$  to every vertex of  $\Gamma$  and a continuous functor  $\nabla \rightarrow \mathcal{C}$  to every arrow of  $\Gamma$ , with matching sources and targets.

**Definition 3** For a topological category  $\mathcal{C}$  and a digraph  $\Gamma$ , we call a gesture with skeleton  $\Gamma$  and body in  $\mathcal{C}$  a morphism of digraphs  $g : \Gamma \rightarrow \vec{\mathcal{C}}$ .

For example, if  $f$  is a morphism in a topological category  $\mathcal{C}$ , we set  $\mathcal{G}[f] = \text{Digraph}/\overrightarrow{[f]}$  for the topos of relative digraphs over  $\overrightarrow{[f]}$ , i.e., of gestures with body in  $[f]$ . Finally, we define

$$X \bowtie Y = \coprod_{f \in X @ Y} \mathcal{G}[f]$$

If  $f : X \rightarrow Y, g : Y \rightarrow X$  are morphisms in  $\mathcal{C}$ , then there is a canonical functor of topoi  $\mathcal{G}[g] \times \mathcal{G}[f] \rightarrow \mathcal{G}[g \circ f]$  and therefore a functor

$$\bowtie : Y \bowtie Z \times X \bowtie Y \rightarrow X \bowtie Z$$

This functor is associative up to isomorphisms and therefore defines *the gesture bicategory of  $\mathcal{C}$* , denoted by  $\mathcal{C}^{\text{es}}$ . With this construction, the morphic half of Yoneda’s lemma would consist in characterizing the functors  $\text{es}$  which stem from composing morphisms in the original category  $\mathcal{C}$ . This would enable us to think of morphisms as being represented by gestures and to calculate all of the category’s operations on the level of gestures. Given that the classical “objective” Yoneda lemma already takes care of the reconstruction of point sets from abstract objects by the transition from  $\mathcal{C}$  to  $\mathcal{C}^{\text{a}}$ , this hypothetical “morphic” Yoneda lemma would give us back the full gestural intuition on the level of  $(\mathcal{C}^{\text{a}})^{\text{es}}$  while working in abstract<sup>8</sup> categories.

#### 4.5 An Answer to Lewin’s Question

The first applications of this general approach have been made on the topological category associated with the topological group  $\mathcal{C} = \overrightarrow{\text{GL}}(n, \mathbb{R})$  of affine isomorphisms on  $\mathbb{R}^n$  [30]. In this context, gestures relate to diagrams of continuous curves in  $\overrightarrow{\text{GL}}(n, \mathbb{R})$ , and these may be applied in the case of a musical parameter space isomorphic to  $\mathbb{R}^n$ . Using this approach to spaces of onsets, pitch, and duration in (loc. cit.), we have discussed some gestural phenomena in tonal modulations of Beethoven’s Hammerklavier sonata op.106. Let us give a short example (discussed in detail in [30]) regarding the rhythmical gesture in the so-called catastrophe modulation in the allegro movement of op.106, measures 189-197, from  $E_b$ -major to  $D$ -major/ $B$ -minor. In this modulation all melodic and tonal orientation break down, the “catastrophe” is manifest in a long sequence of diminished seventh chords with a strong rhythmical force that eventually comes to halt at the end in measure 197. This rhythm stems from the fanfare. It is the hypergesture of this fanfare, which we want to describe, refer to figure 2.

In this perspective, we are viewing the score as being the dance floor of gestures. In our “dancing interpretation”, this defines a hyperhypergesture in the rhythm-space of onset and duration—the event coordinates are shown in the left lower corner of figure 2—described as follows. The gestural construction starts with the first ascending curve and deforms to the second ascending curve. The ascending character means that we address a downbeat, a halting energy. This elementary gesture (first arrow) is deformed to a second appearance (second ascending arrow). This deformation is shown as hypergesture  $\rho$  in the left lower part of the right coordinate system of figure 2. This interpretation is ontologically non-trivial since it creates a continuous transition from the initial note to the second longer one, which amounts to imagining an entire curve of intermediate notes that succeed each other in infinitely near onset times and durations. This enrichment in fact fills out the empty time-space that is not denoted on the score by what in our musical imagination takes place while the

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<sup>8</sup>We have to add that a general category is not automatically loaded with a topology. But there are several ways to do so. The most flexible one is to take any set of functors  $T \subset \nabla @ \mathcal{C}$  and to consider the finest topology on  $\mathcal{C}$  such that all functors in  $T$  are continuous. Denote by  $\nabla @ T \mathcal{C}$  the set of curves for this topology.

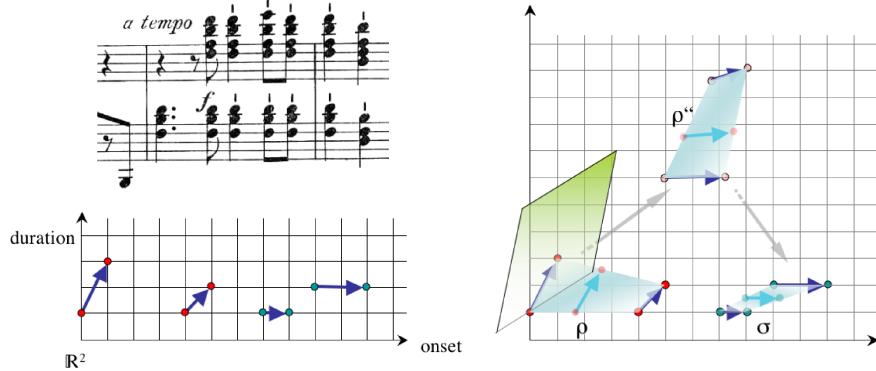


Figure 2: The fanfare in op.106 has a strongly hypergestural shape. Below the score's fanfare, we show the events in the time-related plane of onset and duration. To the right, we see the hypergestures involved in this construction.

first note is being heard/played. The hypergesture connecting the first and second arrow gesture is the connection of this first rhythmic step to the second in the same way, but conceptually and in the perceptive/performative level at a higher stage of imaginative coherence.

The first hypergesture  $\rho$  is followed by a second hypergesture  $\sigma$ , which deforms an arrow connecting two eighth notes to the arrow between two quarter notes. This time, the deformation of these arrows is not the hypergesture connecting a repeated halting movement, but expresses the halting movement of a regular succession of notes of same duration. It is not the repetition of a halting movement, but the halting of a repetitive movement: the roles of repetition and halting are exchanged. In order to connect these two hypergestures  $\rho$  and  $\sigma$ , we give a characteristic gestural transitions: It uses a diagonal mirror operation to flip  $\rho$  into  $\rho''$  in the upper right corner of figure 2. Of course, this requires a complexification of the real 2-space, which is shown by a diagonal mirror to keep the visualization simple. From the intermediate  $\rho''$ , we deform down to  $\sigma$ . So we obtain a hyperhypergesture of skeleton  $\bullet \rightarrow \bullet \rightarrow \bullet$  generating  $\sigma$  from  $\rho$  via  $\rho''$ .

The iterated presentation of the fanfare's first part  $\rho$  in the catastrophe modulation perpetuates that internally already prototyped repeated halting and thereby expresses in an unfolding of the “idea in a nutshell” the failure to release the tension and to modulate in a well-structured way. The final reduction of the initial arrow of  $\rho$  in the ritardando measures 199-200 completes this failure and brings the energies to their exhaustion, the gesture dissolves.

From these investigations we draw the conclusion that such gestures are candidates for Lewin's “characteristic gestures” evoked in his famous statement. The attribute of being “characteristic” is related to the selection of gestures of affine transformations to describe the connection between two collections of notes such as chords, motives, etc. In this discussion, we have enriched the

formulaic language describing tonal modulations by a rich display of gestures, which unwrap these formulas and make the musical drama or dance evident that is casted in the score's process.

## 5 Theory and Experiments

This terminal chapter is somewhat incongruent with the previous status quo styles in that it is not about the mathematical contents of this field of research, but it deals the question of how relevant the mathematical models of music could be to music.

Evidently, this is an important question, but it is more complicated than one would argue at first sight. What is relevance of a mathematical model? Should it help composers to create music, should the model be congruent with a music-theoretical given model, should the mathematical model be congruent to psychological or physiological categories of music perception? Also, as opposed to physics, it is not clear whether we have musical laws or whether we are still in Planck's time of the creation of the spiritual layer of existence, where laws are not built yet. We refer to [28] for a more in-depth discussion of the Planck time topic in music. Is it actually reasonable to ask for such laws? Isn't compositional freedom an absolute barrier against any law?

Despite this set of critical questions, it is also undeniable that compositions are also to a great deal experimental enterprises. A composition can fail, it can prove something, it can be created by applying a very complex set of rules, algorithms or schemes. Music has a great experimental potential. I would qualify the following musical projects as experiments: Schönberg's dodecaphonism, my sonata *L'essence du bleu* [24], the KTH performance rules (Johan Sundberg and collaborators), Pierre Boulez's analyse créatrice, *restructures* by Mazzola, Thalmann, Tsuda and RUBATO®, many of Iannis Xenakis' mathematically driven compositions, Joachim Stange-Elbe's *Kunst der Fuge* performance, David Cope's experiments in musical intelligence, Jan Beran's *Immaculate Concept* [3], etc. But such experiments have also the specification of quality, not everything goes, at least not as a diachronically stable social fact. The communicative character of music is a basic attribute and helps criticize and evaluate compositions and their performances.

In view of these reflections, we want to present the question about the concept of an experiment in music and its theory. Traditionally, one would define it as follows:

**Definition 4** *An experiment is a verification/falsification procedure for a theoretical hypotheses that can be tested in nature.*

Following Immanuel Kant's classical view, an experiment is like a legal action: nature is accused, and the scientist is the prosecutor. This implies that

1. nature is passive

2. it has to behave according to a given formal procedure
3. the prosecutor follows objective laws
4. the judgment's logic is classic: guilty or not guilty.

But music is different, the Kantian concept of an experiment cannot be inherited to music without a thorough critique. Let us enumerate question marks that come up in the context of music:

1. The inner psychological and mental nature of humans is radically different from the external nature of physics. For example, we have language, we use semiotic systems, communicate, use and invent mathematics, etc.
2. The human nature is (much as the physical one) not passive, but reacts differently according to the questions, this is the uncertainty principle.
3. The logic of verification/falsification may be inadequate for the judgment of musical situations. Why can't we use intuitionistic logic as defined by Heyting algebras of subobjects of presheaves, or by topologies?
4. The basic assumption that a "crime has been committed" may be wrong. This means that it is not clear whether we have any musical laws such as the physical laws. Look at the Planck time!
5. It may be a central issue to think about strategies that are adequate to generate new laws. Think of generalized scales, harmonies, counterpoints, or instruments!
6. Is it also possible to make thought experiments in music? Which realities are being thought here?

We do not want to answer these questions, but leave it open to the agents of mathematical music theory to deal in-depth with these questions. They should be answered before we think of a serious methodology of experimentation in music. We do not believe that the question of musical experiments is secondary. We have to be responsible for what we compose and for what we theorize. The dream of a Glasperlenspiel in music (theory) could become a night mare if we do not take the initiative and thematize as soon as possible the dialectic of theory and experiment in music.

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# On D. Tymoczko’s critique of Mazzola’s counterpoint theory

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## Abstract

Some questions and criticism [11] by D. Tymoczko about Mazzola’s counterpoint theory are answered.

## 1 Introduction

Once a writer was asked for the population of Mexico City by a German citizen. When the writer answered “it is about 6 million people, a million more, a million less”, his interlocutor replied “But a million is about the population of my hometown!”. Perhaps as our German friend, we could wish to know the aforementioned figure down to every single individual, but determining this number is an extremely complex labor. For some purposes, it is good enough to count with some approximation, but for many others we need more powerful mathematical tools in order to refine our knowledge.

We can say the same about Tymoczko’s worries concerning the sophistication of some mathematical approaches used by Mazzola to delve into the intricacies of musical phenomena. From his examination of Mazzola’s *The Topos of Music*, he concludes [11] that Mazzola is tacitly asserting “If you cannot learn algebraic geometry [...], then you have no business trying to understand Mozart”. Such a phrase is similar to “If you cannot learn probability and statistics, then you have no business trying to count the population of a city”. Of course you can undertake such a task without having a hint of mathematics beyond mere arithmetic, but the quality of your results would be accordingly limited. Reality (which includes Mozart’s music) is not necessarily simple and easy to grasp, and we can expect that mathematical tools required to increase our understanding of it to be of the same caliber; in particular, in view of Mozart being considered as a genius. With this in mind, we proceed to reply to some of Tymoczko’s statements and questions.

## 2 The reduced strict style

We should start by noting that Mazzola's counterpoint model does not use any algebraic geometry. The model only works with the dual numbers ( $\mathbb{Z}_{12}[\epsilon] = \mathbb{Z}_{12}[X]/(X^2)$ ,  $\epsilon$  being the class of  $X$ ) which are in fact used in algebraic geometry, but their structure does not use anything of that theory. Tymoczko has not read the theory with due precision. However, for the classification of local and global compositions, one needs algebraic geometry, in particular existence theorems of quotient schemes modulo actions of finite groups. And it is here that Tymoczko—to avoid perspectives of plagiarism—would better learn a minimum of algebraic geometry, since he was informed in public during the first conference of the Society for Computation and Mathematics in Music in Berlin 2007 by Mazzola that his orbifold approach to chords as quotients of tone sequences modulo a permutation group had already been realized in Mazzola's local classification theorem in [5, pp. 12, 13].

Tymoczko begins by questioning the simplifications made by Mazzola and Muzzolini in the analysis of the so-called *reduced strict style*, which results from “reducing” modulo octave the Fux's rules in any possible diatonic scale. For the question “why Mazzola permits progressions [within the reduced strict style] from unisons to octaves (or vice versa), while prohibiting the analogous progressions between fifths and twelfths?” the argument given in the paper [7, p. 19] is: Twelfths are not allowed because major tenths are the maximal intervals admitted; the reference to this restriction is Ernst Tittel's classical counterpoint book [10, p. 55]. Therefore, by definition of the reduced style, fifths are the only inverse images of the reduction modulo octave and thus parallel fifths are certainly forbidden in the reduced style. But octaves and primes are both allowed and therefore their identification modulo octave cannot result in a forbidden parallel of primes in the reduced style. Again, Tymoczko seems not having read that argument, or perhaps his reading was not precise. It must be stressed that, while in the reduced strict style the parallel fifths are *axiomatically* forbidden, in Mazzola's model this is *deduced* as a theorem [6, Table O.1.6].

Regarding the “battuta”, the use or prohibition of it is certainly a subtle issue, but there are at least two reasons for not allowing it in the reduced strict style. The first is that Aloysius considers it as an error in his pupil's exercises (even if he leaves to his discretion to follow the rule or not; in any case, we can choose to follow any rules we wish); the second is that Aloysius himself cannot give a full explanation of the origin of the rule, but the mathematical model can. Specifically, Fux states that the example  $(E3, C4) \rightarrow (D3, D4)$  is acceptable but  $(C3, E4) \rightarrow (D3, D4)$  is not, exactly as Mazzola's model predicts.

Something similar applies to repetitions; the phrasing of the four “fundamental rules” in Fux's treatise imply that he does not consider the repetition of a note as an skip or step. In particular, he describes oblique motion as “when one part moves by step or skip while the other remains stationary” [4]. We should also stress that in Mazzola's model the repetition of an interval is not forbidden: it is simply not considered, for it only covers changes of intervals. Hence this point of the model has been misunderstood.

### 3 Real-world practice

Tymoczko provides further evidence for his claims (concerning wrong predictions by Mazzola's model) using examples from actual musical compositions from the Renaissance. Nevertheless, there is a problem exhibiting “deviations” from Mazzola's rules in the practice of Fux, Lassus or Palestrina, because these amount for just a few examples or anecdotal evidence. He should show a statistical account of how many times that kind of “violation” occurs in reality. In any case, practice also deviates significantly from Fux's theory (like using fourths or writing parallel fifths) but this fact is not mentioned by him. In an analysis [9] of first-species passages (with a total of 203 transitions) of the Missa Papae Marcelli by G. P. da Palestrina, the composer infringes on 15 occasions Fux's rules, and in 6 times Mazzola's rules. Even if marginally better (considering that Mazzola's model is nearly as twice more permissive than Fux's theory), the fit of Mazzola's theory provides some justification of its relevance. Furthermore, Fux's theory is supposed to be a reasonable approximation of the practice of counterpoint, so we would expect a closer observance by Palestrina of Fux's rules than of Mazzola's model. The comparison we had drawn suggests that neither of the two theories is better in practice.

We can elaborate on the previous point. In the first place, approximately 13% of diatonic transitions are forbidden by Mazzola's model, whereas roughly 3% of the transitions used by Palestrina in the Missa Papae Marcelli correspond to Mazzola's prohibitions. Analogously, 22.65% of the diatonic progressions are not acceptable in the reduced strict style, but Palestrina diverged from it 7.39% of the time. This fact can be interpreted in two ways: One is that Tymoczko is right and only a minority of Fux's rules are reflected in Mazzola's model; the other is that Palestrina (as a natural process) incurred in Mazzola's prohibitions in a *smaller* proportion as he did with Fux's prohibitions. In second place, 65.15% of the diatonic transitions are explained by one or two symmetries<sup>1</sup>, but in Palestrina's writing this happens in 82.27% of the transitions! This remarkable phenomenon provides further evidence for the musical “anthropic principle” postulated by Mazzola, since Palestrina (of course, unbeknownst to him) preferred the transitions governed by few symmetries more often than statistically expected.

The fact that Tymoczko cannot “imagine a plausible mechanism that would serve to connect Mazzola's mathematics to real-world musical practice” does not mean that such a connection does not exist. The rigidity of the set of counterpoint consonances (which were chosen according to musical practice) modulo octave is a mathematical fact, and practically all of the conclusions of Mazzola's model result from it. Tymoczko never refers to this part of the model, except that he claims that the symmetries of the model are “deep”, which is by no means true for anyone who has ever studied mathematical group theory. He also contradicts “real-world musical practice” with his own use of the abstract

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<sup>1</sup>An admissible progression in Mazzola's model can be explained by one up to five symmetries. For instance, the progression  $(D3, F3) \rightarrow (C3, G3)$  is explained by exactly one counterpoint symmetry.

symmetric group in his orbifold theory. Mathematics is always a foundation of “real-worlds”, in physics and elsewhere.

Tymoczko also omits the EEG experiments conducted by Mazzola, Wieser et al. [8] concerning the neurophysiology of the consonance/dissonance dichotomy and its polarity, which provide evidence that the human brain reacts to the effect of the polarity on a consonance. Again, music theory and music itself is the creation of human beings. What does “reality” means in such a context?

Additionally, the practical dimension of music can be used in multiple ways to support music theory. Reasons from vocal music may have inspired the prohibitions of tritones and parallel fifths, but this argument is not convincing enough since instrumental counterpoint is not limited by these constraints. In this regard, the intuition provided by what Western humans can sing does not tell us whether there are alternative ways of understanding consonance and dissonance or how to extend counterpoint to the microtonal setup. Where intuition fails, mathematics can suggest the path to follow, with the prediction of six distinct counterpoint “worlds” and explicit rules for microtonal counterpoint in Mazzola’s model. These extensions have also been implemented in computer programs by one of the authors [1] and Julien Junod [3]. This is real-world: It sounds, it can be calculated concretely, and its is relevant to composers. Why is it also a real-world concern of theorists? Because the existing counterpoint theory has a weak theoretical basis. Above all, the forbidden parallels of fifths are justified in a psychoacoustic way: perfect intervals are near to fuse. But this is a justification which refers to another ontology, which is not music-theoretical, but psychological. The reference is to just tuning (perfect fifth has frequency ratio 3/2), but the same argumentation would classify the perfect fourth (with frequency ratio 4/3) as being consonant; in fact, this is true in Pythagorean music theory. Carl Dahlhaus [2] has pointed out that it is not psychoacoustics, but compositional theory which could explain the dissonant qualification of the fourth in counterpoint. Our model, refraining from reference to psychoacoustics, gives an answer regarding this problem, and just tuning only creates the contradiction with the perfect fourth as explained above. Our reference to pitch classes is not, as Tymoczko believes, our anachronistic attitude, but a simple mathematical abstraction without diachronic relevance. Embedding a diatonic scale into a chromatic one is not, as it is suggested, a special property of pitch classes; this is well known to be the case for just tuning, too.

It is interesting that the blending of perfect with imperfect consonances, which is generally accepted as a core theme arisen from counterpoint, is not seen by Tymoczko as being dealt with and perfectly modeled with the local symmetries in our model. The guidance we provide for moving between these consonances is a direct consequence of these symmetries. They “internalize” the external dichotomy consonances/dissonances within the consonant half of intervals. The use of local symmetries is explicitly related to their well known role in the creation of tension in physics, so the argument of describing tension by local symmetries is clearly given. And we shall insist: All the allowed and forbidden interval transitions are derived from the model and they are nowhere presupposed. Tymoczko does not understand this difference, and he just repeats

the psychoacoustic argument for forbidden parallels of fifths.

With respect to the interchangeability of voices, there are ways of discerning cantus firmus from discantus; in some compositions the difference is obvious because of the appearance of fourths. In other words: if the distinction between cantus firmus and discantus were pointless, double counterpoint would be trivial. Thus the asymmetry between cantus firmus and discantus is essential for counterpoint, even if it is a necessary hypothesis for an orbifold model. Besides, Mazzola's theory provides a precise mathematical manipulation that permits voice crossings [6, Section 29.6]. Tymoczko cannot cover the distinction between sweeping and hanging counterpoint, for only a hierarchy of cantus firmus against discantus can do this. The intervals in the dual number description are therefore associated with two projections  $\alpha_{pm} : \mathbb{Z}_{12}[\epsilon] \rightarrow \mathbb{Z}_{12} : a + \epsilon.b \mapsto a \pm b$ , addition for sweeping, subtraction for hanging counterpoint. In hanging counterpoint  $a - 7$  is allowed, but in sweeping counterpoint the same pitch class  $a + 5$  is not; these subtleties escape the unordered model.

## 4 Conclusion

Finally, some remarks concerning the “statistical rebuttal” are in order. Let us accept Tymoczko's count of 65 Fux-inadmissible diatonic progressions among the 287 possible, and that only 19 of them are correctly predicted by Mazzola's model. The probability of guessing 19 *or more* prohibitions in 37 trials is

$$\frac{1}{\binom{287}{37}} \sum_{k=19}^{37} \binom{65}{k} \binom{222}{37-k} \approx 3.6023 \times 10^{-5},$$

or less than one in twenty thousands. Thus in 99.996% of a large number of experiments involving a person trying to guess counterpoint rules in 37 trials (with no knowledge of the technique), the individual will stumble upon with 18 *or less* prohibitions. Moreover, his average number of matches would be between 8 and 9.

Considering the number of mathematical models of counterpoint that have been formulated during the last two millenia (which, for sure, are much less than twenty thousands), could we think that one that makes a 29% of correct predictions is bad taking into account that we would expect only a 13% success rate on average? It is true that the mathematical model's predictions are not random, but they could be thought as such since they do not rely on the psychoacoustic or vocal considerations that, allegedly, led to Fux's theory. Note we call a success a concordance with *another* theory, not with actual musical practice. In fact, an agreement close to 100% would be rather suspicious (in particular if the model is almost as complex as the phenomenon we are trying to explain) for it could be an example of overfitting.

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# Computers with personality: a mathematical approach to musical style in improvisation

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## Abstract

We discuss the notions of musical style and personality from a formal perspective. First we recall some important historical facts about them, from early music theorists to contemporary approaches. We present an overview of mathematical tools that might be useful in developing a theoretical framework including formal languages, Markov chains, genetic algorithms, neural networks and finite state machines. At the same time we develop some algorithms illustrating the main ideas. This also includes a program based on a finite state machine that is able to improvise in real time incorporating some basic stylistic trends of the fragments it is presented.

**Keywords:** Musical improvisation, finite state machine, SuperCollider, genetic algorithm, Markov chain.

## 1 Introduction

This science [mathematics] is the easiest. This is clearly proved by the fact that mathematics is not beyond the intellectual grasp of anyone. For the people at large and those wholly illiterate know how to draw figures and compute and sing, all of which are mathematical operations.

Roger Bacon<sup>1</sup>, c. 1265.

In modern society people are exposed to music all the time, even before being born. It is by now a very widespread practice to make a baby listen to this or that composer with the purpose of improving her or his abilities. So we

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<sup>1</sup>As quoted in [5].

might say that our musical personality begins to be shaped very early in our life and that this process continues all the time throughout it. In the case of a musician and, more specifically, of an improviser musician, these influences are particularly important, since they contribute in a significant way to determine her or his style and musical language. In the past few years, several approaches have been proposed to study musical style from a formal or mathematical point of view. Moreover, there are already various software platforms that allow real time interaction among performers (human beings or computers) during improvisation or other software, both commercial and free, that allow for some degree of improvisation or that are used as accompanying (comping) tools. In many cases however, the style, once chosen, is already fixed and remains so (e.g. Band in a Box) or, at the other end of the spectrum, the system reacts *de novo* to the other performer or performers<sup>2</sup>. A central ingredient in improvised music is given precisely by the fact that each performer has a different musical personality and much of the variety, spontaneity and interest in a session comes precisely from the interaction of different people with different backgrounds.

In this paper we present an improvisation system and its implementation (in SuperCollider) in an attempt to formally start exploring questions and issues related to musical personality and style. There could be indeed several approaches. We focus on one that we believe can provide a good basis for further theoretical studies as well as a relatively easy implementation. Very roughly speaking, we first identify the resources and idioms of a performer, assigning them a weight proportional to the frequency with which they are used. We then incorporate both elements (structures and frequencies) to construct a finite state machine which, so to speak, encodes the personality of the system. In order to be able to carry out this program, we need to develop some methodological aspects. In the next section we explain in more detail some tools arising in computer science, specifically from formal languages and finite automata theory.

In section two we present the tools taken from probability theory necessary for our purposes, making special emphasis in Markov chain techniques, while in section three the elements from the area of finite state machines are developed. Finally, in section four and five we briefly discuss other methodologies: genetic algorithms and neural networks respectively. In section six and seven we discuss finite state machines construct a specific system that is useful to illustrate some of the ideas previously introduced. It must be mentioned that each one of these methodologies are useful in their own right and could constitute each one the subject of a separate paper. Finally in section eight we provide a musical perspective and discuss further research and open questions.

However, the purpose of this paper is very concrete and we limit ourselves to present basic considerations about what style from an operational point of view so that we can have a definite starting point. This with the hope that along the way the notion and understanding of musical style and personality will be clarified.

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<sup>2</sup>The platform that a group of researchers has developed with OMax (<http://omax.ircam.fr/>) is a particularly interesting example.

In what remains of the introduction, we discuss different aspects (philosophical, historical, and practical) that we think are relevant before entering more technical issues. In particular, as we just mentioned, we need a more precise and operational definition of musical personality and style.

The approach adopted here is straightforward and we use several methodologies in order to try to capture what the musical style or personality is in a way that leads to a system that is in fact capable of producing music with certain stylistic traits. Of course this presents several drawbacks, oversimplification being one of the most obvious ones. In order to avoid or at least to reduce this danger, we include a brief discussion about what musical style is. An attempt even to summarize the most important aspects or to provide an exhaustive bibliography is not appropriate here (time and space are as usual a handy excuse, but there is another one, namely, that a review paper is currently being written). Nevertheless, we try to point out certain works or ideas that seem especially useful in the context we set up. The problem of defining style is not new and more or less objective definitions are given in many treatises and theory books. Perhaps a landmark in the history of music is of course the work by Zarlino, who summarizes the point of view of what we might call early theorists [9]. However, in many other works an attempt is made to characterize what is considered to be appropriate or “in the style”, e.g. in figured bass tutors or compendia for accompanying singers or melodic instruments. What is more, even in the titles of such works the word “rules” is used (just to mention one, *Reglas Generales de Acompañar* by José de Torres<sup>3</sup> or the rules given by Bach or Heinichen). Throughout the development of western music and other musical traditions, a tentative definition of style reduces to the fact of the existence of a conventionally accepted set of compositional procedures as well as common framework of performance and improvisatory practices. In fact it is curious to observe that the well-known tensions between composers and critics or the general audience can be viewed in many cases as the attempts of the former to broaden and expand this framework against the resistance of the latter. Curiously (or perhaps not so much), the necessity of a more concrete definition of style came up with the arrival of computers to the musical scene. In the context of computer assisted or generated music, the question of writing music in the style of someone is unavoidable. We might say that in a similar way to the traditional definition of style we presented above, in the context of computer music, style can be understood as the collection of admissible algorithms used to generate music. It is important to mention that although this definition might seem more or less the same, there is an important difference. That is the emphasis is made on the procedures rather than on the vocabulary. The point is that this change and opens up a whole discussion on the relationship between style, the creative process and the procedures that a composer might develop as part of her own language. In this respect extensive work has been done and we refer to the books and webpage by David Cope<sup>4</sup> for a detailed discussion as

<sup>3</sup>For links to the original edition see  
[http://es.wikipedia.org/wiki/Jos%C3%A9\\_de\\_Torres](http://es.wikipedia.org/wiki/Jos%C3%A9_de_Torres)

<sup>4</sup>For instance his books [4] or [3] and his webpage.

well as further references. We might rigorously define style (although it might not be very useful or in some cases too restrictive) as a formal language in the context of which composition is taking place (see the next section).

It might be useful to propose a practical test to know whether a certain piece is in the style of a specific composer which is analogous to the Turing test in artificial intelligence. In this test a person is supposed to have a conversation with a computer (via perhaps a human “translator”), which is considered to be intelligent, if the person in question cannot tell if the other interlocutor is a machine. In the same spirit, a composition might be considered to be written in the style of a certain composer if a typical listener (or the average from a representative sample of listeners) cannot distinguish between an authentic composition by the composer and the one under study. These two “definitions” are at the endpoints of a wide range of possible descriptions of style. The first being completely formal or mathematical, the second strictly based on a psychological appreciation. We of course are not looking for the “correct” one and only present them as useful reference and guidelines. Since we defined style in the framework of a formal language, we provide some related basic notions.

## 2 Formal languages and the identification of motives

A formal language is just the mathematical formulation of a language: a collection of symbols (the alphabet) with precise rules (syntax) to form “correct” words, expressions and sentences. For instance, we might look at the genetic code as a formal language in which the symbols are the letters denoting the molecules: A, C, T and G and the rules are given by the corresponding chemical restrictions in such a way that a word of three letters (codon) from this alphabet specifies a certain aminoacid.

## 3 Probabilistic approaches

As a result of the systematic use of random elements in music, it is natural to include probabilistic models to encode the musical personality of a composer. A good beginning for this section is to understand what we mean by probabilistic approaches. We will not use advanced notions, instead we are going to use very simple and classical ideas from probability to make a language that will allow us to describe, approximately, musical style. It should be pointed out that this is by no means new.

In what follows, we will assume that the music we analyze has two main components: structures and frequencies. With them melodies are made. An intuitive approach is to think that the groups that form a musical idea usually are of similar length, they are patterns that come with variations to emphasize and enrich the general structure; in terms of frequencies, the melodic lines are commonly developed in the interval of an octave, sometimes extended to two.

Therefore, our sets of finite states associated to a musical piece are composed by rhythmic figures and melodic intervals. That is we think of a melody as a sequence of transitions that can occur and which are determined by intervals and rhythmic values. By counting the notes and intervals that appear in a musical score, three parameters can be characterized, maybe elementary but useful, about musical style. In the first place, according to the bases of the western musical system, the tonal and rhythmic basic features of the piece. As a second parameter, in a probabilistic sense the proportion with which each state is visited in the sequence of transitions we described above. In the third place, we can find out the dynamical behavior of the piece, i.e. the frequency with which the author moved from a state to another. This data allow to set, later, the transition probabilities between the states in a given musical style. In other words, depending on the style, the use of some intervals and melodic contours are more likely to appear. For instance, in a C major melody, from F is more likely to resolve to an E.

## 4 Musical Markov chains

As a first approach to the probabilistic personality of a composer, it is proposed that the election process of the states that will form a melody, musical motif, phrase or theme, behaves like a Markov chain, i.e. that they only depend on the previous state. Under this assumption, the dynamical behavior of a piece allows us to build, formally using Markov chain theory, the transition matrix and according to the count related to each state that forms such chain -from now on we will refer to it as a Musical Markov chain. Furthermore, one can construct transition matrices that, in addition, consider the proportional weight of the frequency that the composer uses to "move" between states (notes, chords, etc) in a piece. Applying this analysis over representative composers of different styles and genres, musical Markov chains will be a useful mathematical tool for the analysis of the creative musical compositional process. Let us present some more details. In formal terms, we attempt to build sequences of lengths and notes chosen from the fragment in question. These sequences will be considered as Markov chains, i.e. :

$$P(X_n = j | X_{n-1} = i, \dots, X_0 = x_0) = P(X_n = j | X_{n-1} = i) \\ x_0, x_1, \dots, x_{n-2}, i, j \in S.$$

The finite space of states will be defined by two subsets: , the set of frequencies (pitches) and , which include rhythm lengths. We denote the probability of moving from a state to another state, as follows:

$$P_\tau = (p_{i,j}) \quad i, j \in T_\tau \quad \text{and} \quad P_R = (p_{i,j}) \quad i, j \in S_R.$$

Where each state has a probabilistic weight according to the dynamic behavior of the piece and the hierarchy between states. Both parameters are taken

from the analysis of musical scores proposed earlier. It is important to point out that the use some notational system, in this case the traditional western pentagram has a very direct influence in this methodology. Thus, a Musical Markov chain is calculated from an initial distribution:

$$\pi_i = P(X_0 = i)$$

and transition probabilities:

$$p_{i,j} = P(X_n = j | X_{n-1} = i) \quad i, j = x_0, x_1, x_2, \dots, x_{|S_*|},$$

where  $|S_*| = n \leq \infty$  is the number of elements in the subspace of results  $S_*$ . Without loss of generality, it can be assumed that  $S_* = \{1, 2, \dots, n\}$ , for any subset of  $S$ .

The point is that the “personality” of the composer will be captured by the properties of the transition matrix, the weights with which some pitches and rhythmic values are chosen.

## 5 Genetic algorithms

Another important approach that can be used to detect or identify stylistic trends are genetic algorithms (GA). They had been used extensively for several years. The implementation of such methodologies is found, mainly but not exclusively, in quantitative disciplines such as: biochemistry, cryptography, data mining, applied mathematics, physics and so on [7].

GA are based on Darwin’s Natural Selection Theory which states that, within a population, those members who can adapt easier to the constantly changing environment have more chances of surviving and therefore, spread those adaptations through their offspring.

GA are commonly used to find optimal solutions to difficult problems involving large systems. Because GA use probabilistic methods, the solutions found with these techniques are accurate but not exact. Therefore these methods are called heuristic algorithms. Using the traditional approach for finding exact solutions to complex systems may result in systems equations which are unsolvable in practice. This could be avoided using GA in situations where exactness and time constraints can be handled.

In this work we implement a GA to optimize a real time stylistic improvisation based on two components: the surrounding sound and a judging human agent (who could be a musician, composer or indeed any user) who qualifies in real time a sample of notes sequence based on a certain stochastic matrix. Although plenty of variations of GA are in use today, we chose the classic GA methodology, leaving more sophisticated versions for the future.

Any GA needs a vectorization of the problem to be solved. A vector is called a chromosome or genome and it is the basic unit of the GA. It could be compared to a single individual in a living population. One needs to build a population of these vectors. Therefore a population is a set of “almost” identical chromosomes.

In order to evolve the population and reach the desired solution we need variation, just as in nature. All good biologists know that one of the main driving forces in nature is mutation, moreover, unpredictable variation. For our GA to work we need this mutation and translate it properly to the realm of computer science. We need to build an effective way to modify chromosomes. In this work, each an individual is an object with two associated stochastic matrices as attributes. This object has a special method, designed by us, that mutates each matrix. We use a self-organizing perspective based on the classic Self-Organized Criticality (SOC) model proposed by Per-Bak *et al.* in 1988 [1].

## 6 Methodology

As described above, we used two transition matrices based on a discrete Markov stochastic process (also known as Markov Chain Model), one for the melodic component and the other one for the rhythmic component. Therefore the atomic component of the improvisation is a tuple  $(m, r)$  where  $m$  is a member from the melodic state space and  $r$  a member from the rhythm space state. Each entry  $(i, j)$  on the matrix represents the probability of passing from note  $i$  to note  $j$  in one unit of time.

With these probabilities we could simulate sound sequences of states of arbitrary length. These probabilities are calculated with the Markovian methods described earlier.

Once we have determined the stochastic matrices we proceed to build a population  $P$  of  $n$  objects, each with two mutated variants of the original matrices. With the population ready, each member forms a note sequence of the same length and they are immediately presented to the human improviser who qualifies them in a range between 0 – 9. This qualification serves as a fitness parameter, which will be used in the selection method.

### 6.1 The artificial selection method

Each member stores, as an attribute, its respective fitness value. Once qualified all the population members, the algorithm performs a selection method which consists on a roulette technique.

### 6.2 The roulette

Let  $P$  be the population and  $m < n$  the number of effectively reproductive members of  $P$ . This implies that in each generation there would be members that will and will not be in the next generation. This will be based on member's fitness attribute and although each has a positive probability of been selected for reproduction its probability would be proportional to their own fitness. After selecting the  $m$  strongest members we proceed to randomly select  $k = m - n$  and clone them inserting the new ones on  $P$ . This is done in order to maintain the same population size ( $n$ ).

### 6.3 Crossover hybridization

After performing the artificial selection method, a crossover hybridization is implemented. This method consist of taking a pair  $(a, b)$  of reproductive members of  $P$  and form two distinct members by patching the first half of  $a$  chromosome with the second half of  $b$ . The other new member's chromosome is built with  $a$  second half and  $b$  first one. This method is named after nature's own genetic process involved in all eukaryotic cells which is by far more complex. It is consider that on special circumstances crossover will not work, so in order to improve our algorithm we will use a probability of 0.7 of making crossover for any pair  $(a, b)$   $a, b$  in  $P$ .

### 6.4 Mutation with the SOC method

For the final step a mutation method is needed. As it was pointed out earlier, we use a variation of the SOC method described by Per Bak, et.al. in 1988. This method is based on a one dimensional growing sand pile simulation. The algorithm is as follows: Consider  $a$  a member of  $P$  and  $M$  a matrix of  $a$ . Let any row  $x$  of  $M$  be a discrete probability distribution function  $F_x$ . This can be done because the sum of all its elements equals 1. Now, let  $MAX \in \mathbb{R}$  be a fixed value that will be the maximum pile height and  $\epsilon \in \mathbb{R}$ . Randomly choose a state  $y$  of  $a$  (a column or class of  $M$ ) and add  $c$  to its current value i.e.

$$F_x(t+1)(y) = F_x(t)(y) + c.$$

After this step we have two possible cases.

1. If  $F_x(t)(y) + c \leq MAX$  then  $F_x(t+1)(y) = F_x(t)(y) + c$
2. If  $F_x(t)(y) + c > MAX$  then, with probability 0.5,  $c$  could sum to the left  $F_x(t+1)(y-1)$  or to the right  $F_x(t+1)(y+1)$ .

We will consider a cyclic arrangement of states, that is  $F_x(t+1)(y+n) = F_x(t+1)(y)$ , where  $n$  is the range of the space state of  $a$ . Once every member of  $P$  is mutated we ran over the whole process again, from the artificial selection to mutation, until reach the desire improvisation style.

Genetic algorithms (GA) had been used widespread for several years. The implementations of such methodologies are found, mainly but not exclusive, in quantitative disciplines such as: biochemistry, cryptography, data mining, applied mathematics, physics and so on.

## 7 Neural networks

This is indeed a very useful and powerful tool than in principle could help recognize stylistic characteristics in a composition. The method is an iterative scheme that again assigns weights to the connections of a network so that at the end of the process the systems “learns” something. For instance, the network can be trained to distinguish among several patterns. This approach has been

extensively used to classify objects (including sounds). In work in progress we intend to use this approach in order to calibrate the weights used in the Markov chain method described above. The idea is that some other, more subjective, criteria can be introduced in the learning process. Here, for simplicity and also because the example we develop does not make use of neural networks, we limit ourselves to just mention their potential applications.

## 8 Algorithm implementation and Finite State Machines

The algorithm implementation is developed in SuperCollider, which is an environment and programming object-oriented language for real time audio synthesis and algorithmic composition. SuperCollider works with UGens. These represent calculations with signals, which are the basic building blocks of synthesis definitions on the server, and they are used to generate or process both audio and control signals<sup>5</sup>. The algorithm implementation has three steps.

### 8.1 First step

We read the audio input. This allows us to have an interaction between instruments (real instruments in these case) and the computer. For this we use the UGen “SoundIn” which is designed to read audio from the input microphone/line of the computer or soundcard. UGen is based on the UGen “In” to read a signal from a bus. In this case we work only with one bus (mono signals). Once we have the input we use the UGens Amplitude, Pitch, SendTrig and Impulse, to adjust and analize the signal. These are described in the next lines.

**Amplitude:** allows to control the signal amplitude input.

**Pitch:** returns two values; a freq which is the pitch estimate and hasFreq, which tells us whether a pitch was found or not. The UGen is also useful for working with signal’s threshold and adjusting the sensibility of the input.

**SendTrig:** On receiving a trigger the UGen sends a trigger message from the server back to the client. In this case we use two SendTrig in the script. One to indicate when a pitch has been found and other to indicate the frequency found.

**Impulse:** To create outputs non band limited single sample impulses with some frequency.

### 8.2 Second step

Once we have the signal processed we create an Array using the UGen with the same name. And using the Oscresponder UGen which allows us to register a function to be called upon receiving a specific command from a specific OSC

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<sup>5</sup>For references and more details go to <http://supercollider.sourceforge.net/>

(Open Sound Control) address. In this case the Oscresponder receives the messages of the frequency value and then plug them into the Array. The capture of data must be stopped before the analysis.

### 8.3 Step tree, “Creating an improvisation”

When we already have the string of characters (each one of them is a frequency caught) we work with it to create an improvisation. First we use a Dictionary to map the values of the frequencies, each one with a specific label (keys in supercollider). Then a Dictionary is used to create an associative collection mapping keys to values. Now we have each frequency associated with a number (key). These labels will be used to feed another UGen for modeling a Finite State Machine (PfSM).

Finite State Machine (FSM): this is a mathematical abstraction which consist of a group of states “Q” interrelated by transitions between them. The machine has one or more initial states and one final state. The transitions occur when the machine is fed with data strings. In our script the data strings are taken from the dictionary labels and they will be the machine states. Then in the Finite State Machine (Pfsm), every state consists of an item and an array of integer indices of possible next states to go. The initial state can be chosen at random from an array of entry states. The chosen state’s item is returned and the next state is chosen from the new item’s array of possible next states. As we said there is an end state, when it is chosen, the stream ends. Then the machine creates a new string of characters while the states change. This new string of characters will be collected and played by sound generators Pbinds by assigning them the frequency of each state in the string and an arbitrary duration. Here is an example code for Pfsm.

```
//Finite State Machine
// in C maj

(
Pfsm([
#[0, 0, 0, 0, 3, 4], // entry states

//Tonic 1 (== state 0)
Pbind(\dur, Pseq([1/8,3/8,3/8]), \midinote, Pseq([60,64,67], 2)),
#[0, 1, 3],

//e2 (== state 1)
Pbind( \dur,Pseq([2,4,1/2]), \midinote, Pseq([62,65,69]) ),
#[0, 2, 3],
//e3 (== state 2)
Pbind( \dur, 1/3, \midinote, Pseq([64,67,71]) ),
#[1],
//e4 (== state 3)
```

```

Pbind( \dur, 1/4, \midinote, Pseq([65,69,72]) ),
#[2,2,2,4],
//e5 (== state 4)
Pbind( \dur, Pseq([1, 2/3, 1]), \midinote, Pseq([67,71,74]) ),
#[0, 4],
//e6 (== state 5)
Pbind( \dur, 1/4, \midinote, Pseq([69,72,76]) ),
#[4,4,4,4,6],
//e7 (== state 6)
Pbind( \dur, 1/4, \midinote, Pseq([71,74,77]), 2) ,
#[0,7],
//e8 (== state 7)
Pbind( \dur, Pseq([1,1,2]), \midinote, Pseq([72,76,79]) ),
#[0],
// terminal state
nil, nil
]).play;
)

```

## 9 Algorithm

```

//Code of Matautomata I

(
SynthDef("pitchFollow1", { | umbralAmp=0.32 |
var in, amp, freq, hasFreq, out;
in = Mix.new(SoundIn.ar([0,1]));
amp = Amplitude.kr(in, 0.05, 0.05);
# freq, hasFreq = Pitch.kr(in, ampThreshold: umbralAmp, median: 7);
SendTrig.kr(Impulse.kr(10), 1000, hasFreq);
SendTrig.kr(Impulse.kr(10), 1001, freq);
}).play(s);
)

~notasAtrapadas=Array.new

(
var freq;
~notas=OSCResponder(nil, "/tr", {|...msg|
//msg[2].postln;

if(msg[2][2]==1001, {freq=msg[2][3]});
if(msg[2][2]==1000, {if(msg[2][3]==1, {
freq.cpsmidi.round.postln;
~notasAtrapadas=~notasAtrapadas.add(freq.cpsmidi.round);
}}})
}).add
)

~notas.remove

```

```

~arrayin = ~notasAtrapadas.asList.toArray.sort({ arg a, b; a < b });

(
~notas = Dictionary[
{Itonic} -> ~arrayin[0].value,
{IIsuper tonic} -> ~arrayin[1].value,
{{IIImediant} -> ~arrayin[2].value,
{IVsubdominant} -> ~arrayin[3].value,
{Vdominant} -> ~arrayin[4].value,
{VIsubmediant} -> ~arrayin[5].value,
{VIIleading} -> ~arrayin[6].value, // leading tone
{VIIItonic8va }-> ~arrayin[7].value
];
)
(
~maquina = Pfsm([
#[0],
{Itonic}, #[1,1,1,1,2,2,2,3,3,4], //0
{{IIsuper tonic}, #[0,0,0,0,0,2,2,2,2,3], //1
{IIImediant}, #[1,1,1,1,3,3,3,3,0,4], //2
{IVsubdominant}, #[2,2,2,2,2,4,4,4,4,1], //3
{Vdominant}, #[0,0,0,8,8,3,3,3,5,2], //4
{VIsubmediant}, #[4,4,4,4,6,6,6,7,7,3], //5
{VIIleading}, #[7,7,7,7,7,7,5,5,5,4], //6
{VIIItonic8va}, #[0,0,0,0,0,8,8,8,8,8], //7
nil, nil //8
], 1).asStream;

~notasmaqui = ~maquina.nextN(100).collect(~notas.matchAt(_)).reject(_==nil);
~sinil = {arg i; if(i.isNumber, {i})};
~array_final = ~notasmaqui.collect(~sinil.value(_));
)

(
Pbind(
{dur}, Pseq([2,1,0.5,0.25], inf),
{midinote}, Pseq(~array_final)
).play;
)

```

## 10 Musical perspectives and conclusions

The potential range of applications of the framework presented here is wide and at the moment several projects are being carried out related to the basic question we posed at the beginning. Namely, whether it is possible to understand what constitutes the musical personality of a performer/improviser. There might be no final answer and it is clear that much research has to be done related to cognitive processes, strictly physiological aspects and other subjective and objective features of the intricate way in which any individual perceives music.

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# Expanding compositional archetypes in a composer assistant tool

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## Abstract

Musical composition generates symbolic representations (i.e., musical scores) of musical ideas. Such ideas are based on subjective temporal interpretations of auditive events. These events are characterized by their frequency, amplitude and their envelope (which determines the quality of tone or pitch). Such elements, which define the musical characteristics of the musical instruments, are part of the material a composer uses to propose its aesthetic solutions. During this process, a composer can follow a set of implicit or explicit rules to guide his/her preferences and express his/her ideas. Music makes its discourse dynamic by generating patterns containing predictability and unpredictability, expectation and surprises. How can we now make such a game between predictability and unpredictability without an established musical language? From my point of view composition and improvisation concerns the idea of mapping, which deals with the realization of conceptual structures as composition. Algorithmic composition means imagining a process or a structure and then applying a mapping process to transform that structure, which resides in the conceptual domain, into sound, which may reveal the original conception. A famous example of mapping process is found in Pithoprakta's work (1955-56) and Acheorripsis (1957) [8, pp. 17-21]. Gestural mapping occurs when a sonic result is achieved by an act of interpretation, whether it be reading a score and/or improvising. My approach in this paper is the configuration of a composition tool, which involves finite state machines, group theory using cyclic groups and dynamic systems theory using chaotic functions. As an example, I will present Lutosławski's procedures and mappings expressed in the musical analysis and studies done by Steven Stucky, Martina Homma, Adrian Thomas and Charles Bodman Rae. The pieces I chose as a reference for building this example were *Jeux Vénitiens*, *String Quartet* and *3rd Symphony*. The five topics I develop with this system will be divided into sections: Section 1 exposes a harmonic conformation. Section 2 explains the use of microrhythm and rhythm generators. Section 3 discusses melodic lines generated by shape interpolation combined with rhythm generators. Section 4 extends embellishing techniques and harmonic blurring, and section 5 gives conclusions and future research directions.

## 1 Harmonic conformation

The systematic procedures in Lutoslawski's interval explorations and his obsession to find a set of notes capable of making natural transits in different transpositions, inspire me to create a rule oriented computer environment mechanism to generate 12 tone pitch chords. My computer environment generates the following characteristics:

1. Sets of chords from 2 to 11 intervals.
2. Its relationships between their complements and/or aggregates.
3. Transposition and pitch distribution for a particular set to be used for a desire orchestral color.
4. Reorganization of subsets of notes once a twelve tone chord is chosen according to composer's rules.

I choose Forte's set theory as a starting point to generate 12 tone chords, because it is an application of mathematical set theory to pitch-class sets (groups of pitches) in music. The relevant parts of his theory identify and classify all possible combinations of pitch-classes (pc) in the twelve-tone system, and it is an entity for determining relationships between various pitch sets. These sets are analogous to chords in traditional theory, but are really any group of notes. The theory also provides ways to determine the interval (class) content of a set if one set is contained in another (the subset relation), and the relationships between sets having the same number of pitch classes (similarity relations). It also embraces all traditional chords under the same theory. Each pc (pitch class) in the twelve-tone system is assigned a number:  $C = 0$ ,  $C\# = 1$ ,  $D = 2$ ,  $D\# = 3$ ,  $E = 4$ , ...,  $A\# = 10$ ,  $B = 11$ . Numbers are also used to represent intervals in semitones. A minor third, for example, is 3 semitones, or just 3. A perfect fifth is 7, etc. I capture the interval structure with a collection of classes I made for the program environment Super Collider [4]. The class `tipo_pc` receives a collection of intervals as input, and returns either another list of intervals as a pitch class set (pcs) or the collection's label in Forte's notation (fnot), using the same principle already developed in Open Music. Both results are given in prime form as described below.

```
pcSet
:: input (11 0 5 6)
pcs-prime-form output => (0 1 6 7)
or
fnot  pcs-prime-form output => 4-9
```

In the following diagram I explain an example of how these three goals were achieved with any interval collection as input.

```
// Array input [2,4,6,8,11,0]

a = Escamol.pcSet([2,4,6,8,11,0]);
```

```

a = [0,1,3,5,7,9]

a.fnot

// returns 6-64

// This message produce an output of interval to that pc-form

b = a.comple

b = [2,4,6,8,10,11]

b.fnot

// returns 6-34

// the message pcTransp enables intervallic statement in symbols
// or integers to be transposed in accordance with the attached
// to the transposition of pitch class sets

c = a.pcTransp

d = b.pcTransp

// pitch distribution by rule

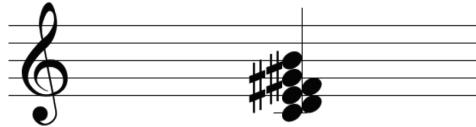
e = Escamol.new

#f, g, h = e.pcAtasca(c,d)

// f instantiates the resultant transposed component, g
// resultant transposed pcs and h instantiates the
// resultant 12 tone chord

```

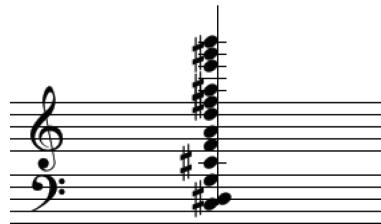
In this example, the input chord is (2, 4, 6, 8, 11, 0):



The pitch distribution rule receives the transposition pcs collection of pitches. The rule classifies the list in odd and even occurrences and returns a new distribution by transposing all odds an octave lower.



An analogous procedure is made with the complement two octaves higher. As a final stage, the program returns three collections of chords: complement, input and 12 tone chords all in transposition pcs and rule distributed. The following is an example of the twelve tone result.



The flexibility of this routine allows the user to experiment and discover different pcs complement combinations and permits the user to redefine the pitch distribution rule according to his/her convenience.

## 2 Rhythm

The compositional separation of pitch organization and rhythmic-durational ideas was certainly crucial for a comprehension of Lutosławski's working methods. According to the articles by Adrian Thomas, Martina Homma, Steven Stucky and some specific passages from *Jeux vénitiens*, *String Quartet* and *3rd Symphony*, I capture enough material to develop an algorithm that could match Lutosławski's procedures more closely. The sketches presented by Martina during the seminar and Adrian Thomas [2, pp. 211-243] finished my enlightening for the design of this part of the tool. I design then, three rhythmic generators: (1) a combinatorial environment for microrhythmic organization, and (2) two rule oriented environments.

### 2.1 A combinatorial environment for microrhythmic organization

This environment should have the capability to unfold any harmonic material containing different rhythmic cells in a canonic fashion with or without aggregates. It should also be able to use any other procedure defined by the composer to manipulate the rhythmic cells (e.g. inversions, prolongation, variable dura-

tion in a specific cell, etc). The goals of this section of the program were to produce:

1. A symbolic representation for each rhythmic cell despite the number of elements of each representation
2. A rhythmic cell distribution in 1 to  $n$  voices subordinated to a certain pattern (e.g rotations, permutations, etc.).
3. A pitch distribution of the rhythm resulting from point 2.

I first design a function consisting of 3 modules. The first receives 3 inputs: a list of rhythm cells (rcs), the cell position, and a cell or group of cells to be substituted. The outputs are: a collection of symbols (csy) and new rcs (if there was any substitution). The input of the second module defines how many voices and the pattern each voice has to follow within the rcs as output (vp). The third module receives the csy, rcs, vp and the chord to be unfold as inputs and returns a score format sketch as output.

## 2.2 Unfolding

To confirm the proper behavior of the program, I first started with 5 simple rhythmic cells Lutosławski used in Symphony No. 2. The pitch material collection for this experiment was (0, 2, 3, 5, 7, 9, 10) and corresponds to a fn 7-35. The program realized the unfolding material the same way that Lutosławski had done in his sketch VIII.2, as shown in Martina Homma's article [2]. Dr. Martina verified that conclusion during her visit to Professor Stucky's seminar. The next graph presents how these goals were accomplished and the resulting score.

```

(
~esc = Escamol.new;
~ryt = Rit_esca.new;
~de = Densidad.new
)

(
var veces, sindiv;

// rhythm cells
sindiv = [1 ! 5, [-1,1,-2,1,-2,1], [-1,1,-1,1], [-1,1,-1], [-2,1]];

// Pattern to unfold
veces = Pseq(sindiv,4).asStream.all;

// symbols assigned by the program for each rhythm cell
~lutly = ~ryt.lutrhy_ly([4,3,4,3,4,3,4,4,3,2,5],veces,[36,48,60,72,84].reverse,24);

ArchLily.genpoly(\lut_ly1,5,~lutly,val:8); //5 equals to the number of voices
LilyOutput.viewPdf(\lut_ly1) //lilypond file format result
)

```

**Unfolding**

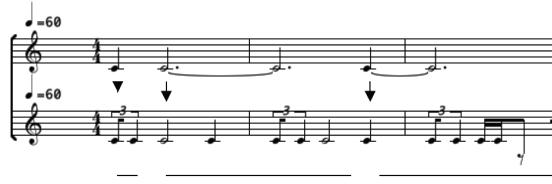
The musical score for "Unfolding" is composed of three systems of music for two voices. The score is written on three staves, each with a treble clef and a key signature of one sharp. The first system has a tempo of 54, indicated by a "54" below the staff. The second system has a tempo of 32, indicated by a "32" below the staff. The third system also has a tempo of 32, indicated by a "32" below the staff. The music features various note heads, stems, and rests, with some notes having vertical stems and others having horizontal stems. The vocal parts are separated by a vertical bar line.

### 2.3 Rule oriented environment

For this section of the program, I designed three inference engines containing a domain for rhythmic cells and some sets of rules that constrained the results. These engines use the classes `Escamol` [5], `Rit_esca` and `Densidad` for Super Collider. In the classes the elements in the domain are the smallest rhythmical building blocks in the system. An element can be either a note value, a pause, a rhythm motif or a time signature. Different strategies are possible when searching for a rhythm sequence. If the user defines the elements as rhythm motifs, he can already at this point have some control of the rhythmical language. The motifs will be the base for the rhythmical language. Their length and degree of difference are important for the result. If the user only gives the system single note values to work with, his rhythmical language will rely more on the program rules. The rules the composer defines will tell the system how the motifs/note values can be arranged and connected to each other. The program allows work on hierarchical rhythm: i.e. some events are given higher importance than others. The hierarchy can either be a metrical hierarchy, or any other type of hierarchy the composer can imagine (for example harmonic rhythm). The composer can use several predefined rules from the se classes. However, an important feature of the program Super Collider is the possibility for the composer to define his own rules expanding and/or creating new classes. The first inference engine is a simple 3 rule object. Its database source is two sixteen and one quarter motives. The rhythmic result is subordinated to the time signature as shown below.



The second engine uses a more complex set of rules: dynamic database motives, fixed number of notes and no repetition between motives. Resultant motives should be subordinated to specific note values as shown in the following sample score.



In this case the number of elements was 12. The rhythmic result had to fit in the values presented in the upper staff. The database consists of 5 pairs or rhythmic cells: (half- quarter), (quarter - eighth), (eighth - sixteenth), (sixteen - sixteen), (triplet eight - quarter triplet)

The third engine contained in the class `Densidad` includes a set of stochastic and probability functions.

### 3 Shape Interpolation

The shapes in Lutosławski's sketches were the main source for building this application. The challenge was to conserve this bifocal principle between rhythm and pitch organization and achieve a coherent linear music discourse. In the next example I use the same type of filtering used by Lutosławski at the end of his 3rd Symphony. As transition material, I create two rows of 9 notes each. The pitches mark what will be the contour lines that will be interpolated in the model shown in the next figure.

Score 1



Score 2



In order to get a smooth transition between the two contour lines the program uses an interpolation function, which gradually changes the contour line from the first to the second rows above. The user is able to define the number of steps. In this example, I choose 24 steps. The group of 24 contour lines is filtered through the `~tradLy` message in such a way that it will maintain the tonality despite register. For the rhythmic interpretation of these contour lines I used the first inference engine described in section 2. An interesting feature of the rhythm engines one and two, not mentioned in the previous section, are heuristic rules. Heuristic rules are neither true nor false, but give different weights to different answers. Musically, this type of rule lets the user tell the system things such as "I prefer an answer with fast note values". In addition I am using Poisson equation to define density in the musical discourse result.

```

~fun = { arg list, int= 10;
var uno,dos, fu;
uno = [[61,62,64,66,69,78,69,74,60]-12,
       [69,66,64,63,65,72,75,68,78]+12].interpolate(12,loop:false).round.flat;
fu = {[]no}
var ed , be;
ed = Ppoisson(8,uno.size).asStream.all.collect({|i,ndx| if(i == 0,{ndx})}).reject(_.isNil);
be = (uno + no).collect({|j| ~ryt.tradLy(j)});
ed.do({|k| be = be.insert(k,\r)}); 
be
};
dos = list.collect({|l| var te = fu.value(l),
                    [te, Pclump(Prand([4,5,6],inf),Pseq([1],te.size)).asStream.all]});
dos.flop
}

(
a = ~fun.value([12,0],8);
ArchLily.genpoly(\interp,2,a,val:8)
)

```

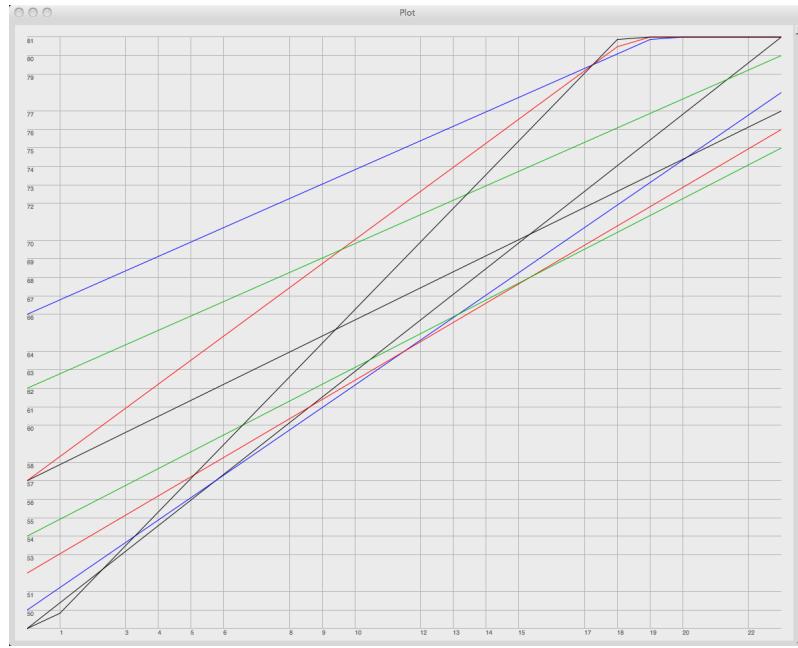


Figure 1: Interpolated notes (represented in MIDI numbers).

**Interpolation**

1      2      3      4      5      6      7      8      9      10      11      12      13      14      15      16      17      18

In the following score I used three voices in which the notes are the same but the heuristics in the rhythm are different. We can perceive some differences from Symphony No. 3, but the sonic result is quite similar in level of complexity and is extremely natural. In the following graph I show the code that generated the sample score entitled “lut-clone”.

```
// fl part
~ciclo = [[60,64,66,67,69,72],
           [60,65,74,79,69,72].reverse + 7].reverse.interpolate(8,loop:false).round;
~ciclo = [[60,64,66,67,69,72],
           [60,65,74,79,69,72].reverse + 7].reverse.interpolate(8,loop:false).round -3;
// -- ciclo para efectos

~ciclo = Pclump(Prand([3,6,8],inf),Pseq(Array.fill(96,{24.rand}) + 64)).asStream.all

~meli = ~ciclo.collect({|i| ~esc.esca(12,5,i)});

// 2 edos mas antes 6
//a = ~ryt.dist_rhy4(90,5.1,4);
)

// checando a escamol
~meli = Array.fill(30,{~esc.esca(6,5,[60,64,66,67,69,72])})

//----- we group with ~empr to generate rhythms -----
//--- flute
(
~mel = ~meli.flat.reject(_== \r);

// cada grupo escamolero lo subdividimos en tuplets
a = ~mel.collect(~empr.value(_));
a = a.flop;
// resultado
b = [a[0].flat] ++ [a[1].flatten(1)];

l = ~mel.collect({|i| ~ryt.tradLy(i)});
)
//clarinet

~mel = ~meli.flat.reject(_== \r) - 3;

// same density as percussion

(
e = Ppoisson(1.5,~mel.size).asStream.all.collect({|i,ndx| if(i == 0,{ndx})}).reject(_.isNil);
e.do({|i| l = l.insert(i,\r)})
)

//-- checamos
MakeLilySequence.perc(l,b[0],b[1]);
(
~mel = [l] ++ b;
ArchLily.gen(\pi_cl_fin1,~mel)
```



## 4 Embellishing techniques and harmonic blurring

After completing the progress demonstrated up to section 3, I decided to continue and try these tools on bigger structures. For this next experiment I again borrowed from Lutosławski's rotation and aggregates principals. I expanded the chord generator to the point where it is capable of producing any number of chords and rotating them in an asynchronous way with transpositions. In addition the generator contains a 12 tone chord dynamic filter. Later, at an editing facility the user can determinate the final touch in register and number of voices. The next step, was to design a melody line generator capable of operating between the chords with a function that I create to do multiple interpolations dynamically between 2 or more elements in a specific pitch group or key. The function also responds to graphic figures similar to the model shown in section 3. The number of voices is determined by the number of notes contained in each chord. I maintained the tonality feature in the melodic generator while transiting from one chord to another and using rhythm engine two as a rhythm generator. Taking full advantage of its heuristic possibilities, the program was generating phrases with different lengths, thus creating a quasi Chain principal, most like a blurring harmony effect. In another experiment I added a rule to maintain the rhythm result in equal lengths along the phrases. The result was an embellishing effect. In this paper I am including the scores of Lut-clone, a rule base chord version structures for Chord\_rhyth\_gen\_2.

## 5 Conclusion

A domain or universe conformed in a collection of procedures and abstractions is my favorite definition for style in music composition. I understand these procedures as the result of a learning process with a systematic methodology which includes: traditional composition techniques mastered in previous centuries, a coherent methodology with a strong philosophical background, aesthetic proofs product of these ideas and finally, pieces of art that synthesize and reflect style

transparency. Finding a tool to precipitate and mature these ideas represents new and refreshing fields for development and exploration.

## 6 Future research directions

I will implement the chain principle by adding rules, which will keep common pitches between two or more different chords simultaneously. The search algorithm efficiency needs to be improved. I am perfecting the aesthetic results, and they will be part of the piece Donaji that will be premiere in Mexico on 2012.

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# The RUBATO®Composer software concept for learning advanced mathematics

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## Abstract

Category and functor theory serve as a framework for the concept architecture of the software RUBATO®Composer. In particular, this software works with components called *rubettes*. To use the majority of these rubettes requires a certain mathematical sophistication. In addition to what RUBATO®Composer is designed to be for the composer and music theorist, it is also an excellent tool for learning sophisticated mathematical concepts. On the other hand, rubettes are being developed that ‘hide’ the mathematics, even presenting interfaces that can be carried out through gestural mouse commands, based on mathematical gesture theory. These high level rubettes could be “uncovered”, so that the music student can really begin to relate to the mathematics on which modern mathematical music theory builds its edifice. At the same time, the mathematics student can witness an implementation of hitherto abstract concepts, and the computer science student encounters category theory in the development of the software itself. The present paper develops these ideas from a pedagogical, mathematical and philosophical perspective, and reports on the case of a student and his experience with RUBATO®Composer.

**Keywords:** RUBATO®Composer, category theory, mathematical music theory, computer-based learning, interdisciplinary curriculum.

## 1 Introduction

Interdisciplinary activity is always a contentious subject. For some it implies possibilities of knowledge expansion and new applications, while for others it signals the danger of superficiality, contamination and surrender to fashion. The position presented here has been conceived in the context of mathematical music theory. On the following pages, the argument will be in favor of (what this author considers) the tremendous possibilities that mathematical music theory, and its applications, have to offer to the knowledge base of mathematics, music, and computer science students, without excluding those from other fields. This does not preclude recognition of the need to proceed with caution. Saunders MacLane [9], one of the creators of category theory, warns about tendencies to underplay proof in mathematics education, as a consequence of the discovery

and intuitive justification of important mathematical results by physicists. It is important, when exploring the development of interdisciplinary curriculum, to acknowledge the specificities of each of the disciplines involved, and to recognize them in the delivery and evaluation processes. This work presents a proposal that considers the following elements. RUBATO®Composer is a GPL software<sup>1</sup>, whose development is based on the category-theoretic concept framework, using a functorial approach. The data format of denotators is based on set-valued presheaves over the category of modules and diaffine morphisms [13]. The mathematics involved is sophisticated, and could be accessible in a formal way to the average mathematics major in the United States in their senior year, after having some experience with courses such as linear algebra, modern algebra, analysis or topology, but would usually be taught at the graduate level. On the other hand, the design of this software, its plug-in components called *rubettes*, its application written in portable Java, and graphical user interface (GUI) possibilities, combined with the aspects of object oriented programming, are of interest to computer science students. Finally, the motivation behind the use of the sophisticated mathematics and software development in general, is the production of a concept format for the musician, composer or theorist, which permits smooth navigation through the knowledge spaces of the multi-faceted field of music [11]. The creation of interdisciplinary curriculum materials and courses, using RUBATO®Composer as a common ground, opens a realm of possibilities for mathematical music theory, and for the development of research and researchers in the field. It also can be justified, in and of itself, where RUBATO®Composer is conceived of as a learning tool.

## 2 Computer-Based Learning for Advanced Mathematical Concepts

Dubinsky proposes that “although using formalism to construct meaning is a very difficult method for students to learn, it may be that this is the only route to learning large portions of mathematics...” [4, p. 1]. He suggests that the writing of computer programs to express mathematical concepts can be an effective way of achieving this goal of advanced mathematical learning. It has been generally acknowledged that there is a gap between the formality of modern mathematics as conceived and taught by trained mathematicians, and the mathematics that are seen by non mathematicians as relevant. When the mathematics are embedded in different practical contexts, it is often easier to get students to think mathematically in a natural manner. ‘Even mathematics students themselves often have difficulty in making meaning out of the formal presentation of their subject’. (MacLane, 2005). A revision of articles from *The International Journal of Computers for Mathematical Learning* during 2006-8 (for example [1, 7, 14, 2]) illustrates that the notion of using specific software to enhance the learning of advanced mathematics has a respectable recent history,

---

<sup>1</sup><http://www.rubato.org>

and has been analyzed using well documented research paradigms. There is a gamut of mathematical subjects touched upon in the articles referred to above. The themes range from representations of infinity through computational construction, the generation of complexity from limited instructions, and embodied semiotic activities related to motion graphs, to mathematical string sculptures, among others. The RUBATO®Composer software opens up the possibility of creating meaning for the formalisms of advanced mathematical areas, and accelerating processes of learning and understanding. These mathematical areas (abstract algebra beyond group theory, category and topos theory) are not really addressed in the literature on computer-based learning, or on collegiate mathematics education in general. Computer-based learning in music is usually related to training in aural skills [3], sight reading [15], and other subjects essential to the music student. On the other hand, there are musical representation languages such as Common Music [16], OpenMusic [1], and Humdrum [8], for composing and analyzing music, that do require programming skills. However, RUBATO®Composer offers the opportunity of introducing the music student, especially one who is interested in the theoretical possibilities of analysis or composition, to the higher mathematics involved in modern mathematical music theory. This can be done in a relatively (not completely) “painless” manner, as compared to what it would require to learn this material in the traditional way.

### 3 RUBATO®Composer as a Learning Tool

RUBATO®Composer is based on the data format of forms and denotators. Forms are mathematical spaces with a precise structure, and denotators are objects in the form spaces. Category theory is the mathematical foundation on which this conceptual basis of mathematical music theory is built. Modules are mathematical structures, similar to vector spaces but with scalars in rings, so that every vector space is a module, but there are many modules that are not vector spaces. However, the module structure preserves and enhances the implementation of many of the most efficient algorithms, based on matrix theory and linear algebra, which have been developed in the computer science field. In the RUBATO®Composer architecture, modules are a basic element, much like primitive types in programming languages. The recursive structure of a form, if not circular, will eventually “stop” at a *Simple* form which, for all practical purposes, is a module. Morphisms between modules (changes of *address*), are built into the software. At the same time, in the development of the data base management systems, the objects must be named and defined in a recursive way and they must admit types such as limit, colimit, and power. In other words, it is necessary to work with the algebraic structure of modules, yet form constructions whose prototypes are found in the category of sets. This is the reason why, in the context of RUBATO®Composer, the approach is to work in the functor category, which consists of presheaves over modules (whose objects are the functors). The development of this theory and its relation to

RUBATO® Composer can be found in [11, 12, 13] .

Through the creation of denotators, and the recursive structure of types when working with forms, the mathematics student accustomed to the formalism of abstract mathematics (in particular, abstract vector space theory, module theory, category and topos theory), has an unusual opportunity to participate in a concrete implementation of these concepts. On the other hand, the mathematics student who still struggles to find meaning in the abstract formalism, may find a vehicle through which this process can be accelerated. Dubinsky [4] has expressed that “... writing, running, and reflecting on computer programs to express mathematical concepts can be an important mechanism for achieving this”, where *this* is the construction of meaning based on mathematical formalism.

The majority of rubettes available at this time are of low level nature. One of the objectives of the developers of RUBATO® Composer is to create more high level rubettes that present ‘friendlier’ interfaces and language for the non-mathematical user, in particular the composer or musicologist. However, even with the high level rubettes that are, and will be, available, it is possible to retrace the steps and uncover the mathematics behind their construction. When the terminology changes from ‘transposition’ to ‘translation’ and, in general, from the musical ‘inversion’, ‘retrograde’, ‘augmentation’, to the language of mathematical transformations, or morphisms, the student is presented with an opportunity to develop an understanding of the formalism behind the meaning.

The *Wallpaper* rubette [17, 13] is a very good example of this ‘intermediate’ type of rubette. Behind its implementation is an application of the mathematical machinery contained in the concept framework. The musical purpose of the *Wallpaper* rubette was motivated by an initial application involving translation in the *Onset-Pitch* plane. This application took place in the *presto* software, an antecedent of RUBATO® Composer. However, in RUBATO® Composer not only translations, but general affine morphisms as well, can be used to generate musical ornamentation. The *Wallpaper* rubette also opens the possibility of generating morphisms in any  $n$ -dimensional space (for example, using the five simple forms of the *Note* denotator –*Onset*, *Pitch*, *Duration*, *Loudness* and *Voice*– an affine transformation in 5D can be defined. This will be explained in the example given later in this paper). When working with affine transformations in 2D space, the command can be given by just ‘dragging’, instead of defining the morphisms. A unit introducing the basic concepts of linear algebra, group theory, and geometry needed to understand mathematical music theory, as it has been developed over the last 40 years, can be created. Students, especially non-mathematics majors, could not ask for a more motivating and relevant entrance to these subjects. The most ‘extreme’ example, up until now, is the *BigBang* rubette, developed in the context of mathematical gesture theory and computer semiotics. It is based on a general framework for geometric composition techniques. Given a set of notes, their image is calculated through affine invertible maps in  $n$ -dimensional space. There is a theorem [11] that states that the affine invertible map in  $n$  dimensions can be written as a composition of transformations, each one acting on only one or two of the  $n$

dimensions. The component functions (the ones that act on only one or two of the  $n$  dimensions) are represented geometrically as five standard mathematical operations that have their musical representation:

1. translation (transposition in music),
2. reflection (inversion, retrograde in music),
3. rotation (retrograde inversion in music),
4. dilations (augmentations in music),
5. shearings (arpeggios).

On the other hand, there is a mathematical theory of gestures which defines a gesture as morphism, and formalizes gestures as curves in a topological space. Through this theory the *BigBang* rubette was designed. It is a rubette in which transformations can be carried out by using *gestural mouse commands* [12]. This is another example of a high level rubette, whose underlying mathematics, rich in both geometry and formalism, can be ‘unfolded’.

## 4 Student Experience

A report will be given of an antecedent with RUBATO®Composer as a learning tool in the case of a mathematics student. The task consisted of creating a form broad enough for the majority of simple electric bass scores, and a denotator which represented the jazz song “All of Me”. The denotator was to represent the 64-note bass line as a “point” in a theoretical basscoreform<sup>2</sup> - which the student also constructed. The denotator would eventually be played (implemented) through the *ScorePlay* rubette. The tasks and procedures required are ‘low level’, where the student must define everything explicitly, from the basic module morphisms underlying the denotators to the matrices that ‘spell out’ the transformation. This is exactly what would be expected when working with mathematics students. The music and humanities students have the option of higher-level procedures, using graphical and even gestural inputs, although it cannot be discarded that, eventually, they too will manipulate the lower-level rubettes. The recursivity of the mathematical definitions of form and denotator are made evident in this application. Figure 1 depicts the complete “basscoreform”.

The name of the form is *Basscoreform*, its type is Limit and its coordinator is a list (product) of forms.

Following the recursivity built into the theory, the student chose one of the list, which in this case was *Generalnotes*. *Generalnotes* has type *Power*, so its coordinator is another form, whose name is *Generalnote*. As *Generalnote* is a form, it has a type which in this case is Colimit, and its coordinator is another list (coproduct) of forms. From this coproduct, the student chose *Simplenote*, whose type is Limit and whose coordinator is another product of forms. In this

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<sup>2</sup>Forms and denotators are defined in [11]. Forms and denotators provide a means for implementing an important part of computational category theory. Forms are a generalization of the concept of data types, and denotators are mathematically defined “pointers”.

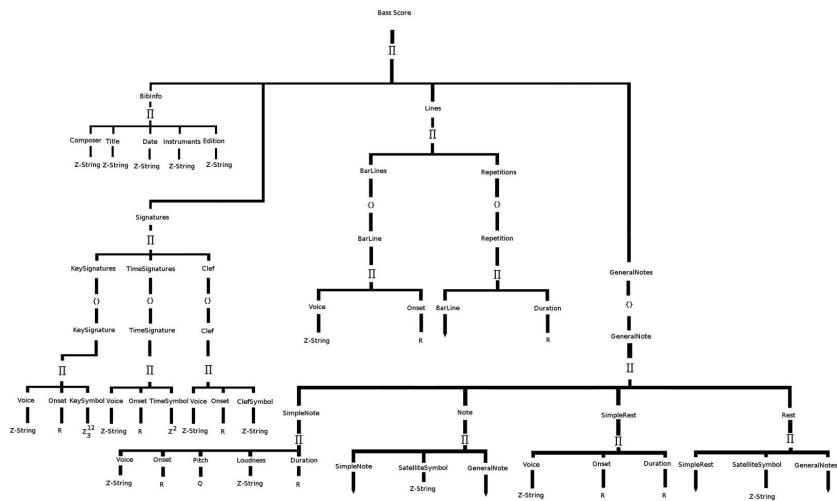


Figure 1: Basscoreform.

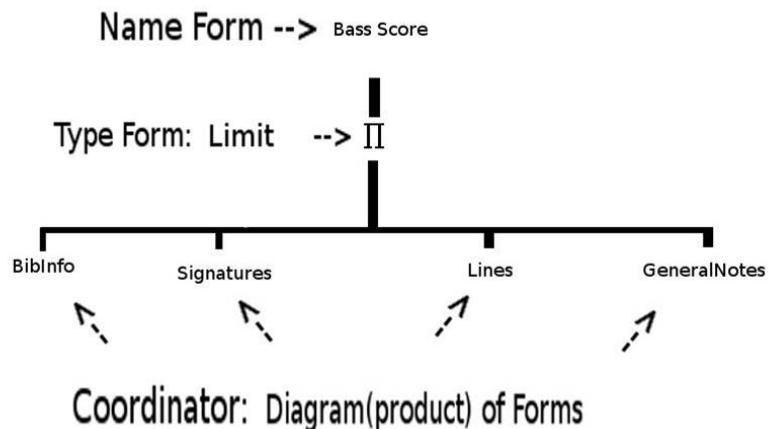


Figure 2: First Example of Recursivity.

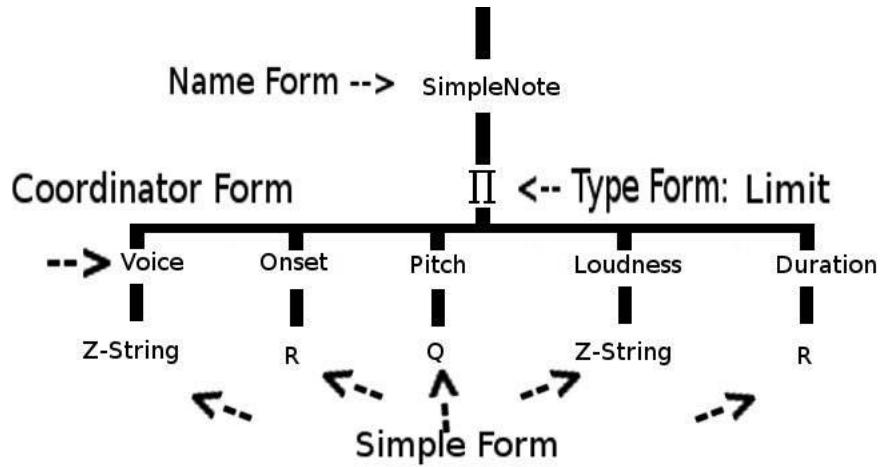


Figure 3: SimpleNote.



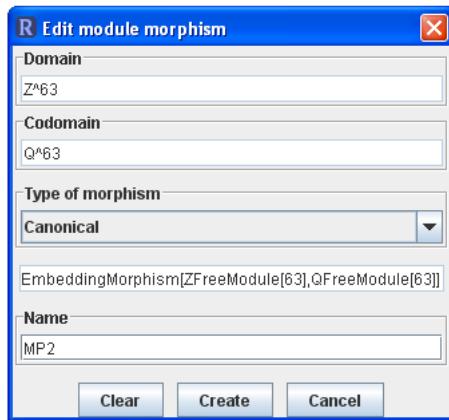
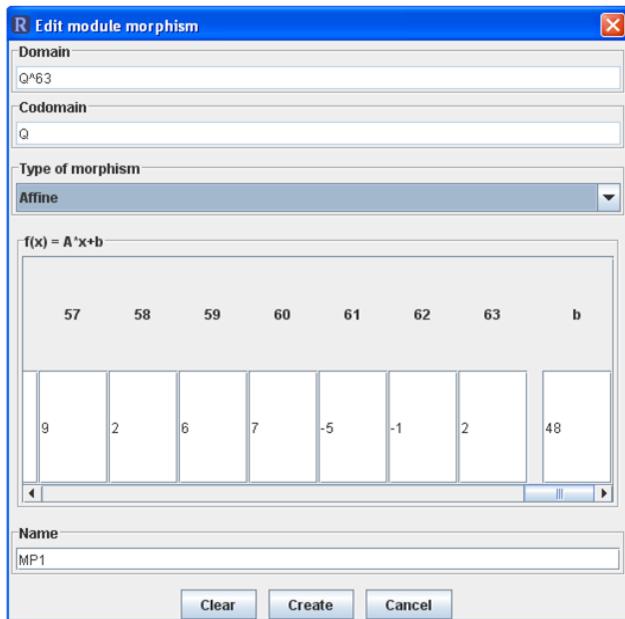
Figure 4: Bass score.

case, the forms are Simple, which means that they “end” in a module. The simple forms of the *Simplenote* denotator - *Onset*, *Pitch*, *Duration*, *Loudness* and *Voice* have coordinators (real numbers, rationals, reals, integers) respectively.

A single denotator  $N$  of the form *SimpleNote* was created from the coordinates of the denotator which themselves are forms of type Simple: *Onset*, *Pitch*, *Duration*, *Loudness*, and *Voice*. In particular, an example of how to build the pitch module morphism “ $mp$ ” is given. All the others are built analogously. To begin with, the module morphism builder in Rubato Composer is opened to start creating a module morphism for pitch. First “ $mp_2$ ” and “ $mp_1$ ” are created, and then they are used to make “ $mp$ ”, which is a composition of the two. For  $mp_2$  the domain is determined from the number of musical notes in the bass line. For instance, the bass line for “All of Me” contains 64 notes (Figure 4) and the first note is established as the anchor note, so the domain is  $\mathbb{Z}^{63}$ . The embedding is evaluated in  $\mathbb{Q}^{63}$ .

Then, the affine transformation  $mp_1$  goes from  $\mathbb{Q}^{63} \rightarrow \mathbb{Q}$ . The module morphism is set up in the same way as  $mp_2$ , but *affine* is selected instead of *canonical*.

To create  $mp$ , the student went to the module morphism builder, and created a module morphism with the domain of  $mp_2$  ( $\mathbb{Z}^{63}$ ) and the codomain of  $mp_1$  ( $\mathbb{Q}$ ).

Figure 5: The creation of the embedding morphism  $mp_2$ .Figure 6: The creation of the affine morphism  $mp_1$ .

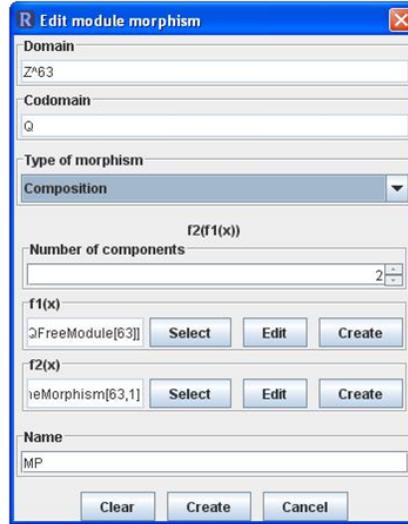


Figure 7: The creation of the composition morphism  $mp$ .

The student then went to the Denotator Builder (Figure 8) to create the denotator Simplenote (Figure 9).

As can be seen, the creation of the pitch morphism is the composition of an affine morphism and an embedding, which represents the 64 pitches of the bass line. This is a module morphism in the category of all left modules over associative rings with unitary, whose morphisms form the set of diaffine transformations. There is quite a bit of abstract mathematics from module, ring and category theory implicit in this last sentence and in the creation of this module morphism ( $mp : \mathbb{Z}^{11} \rightarrow \mathbb{Q}^{11} \rightarrow \mathbb{Q}$ ,  $mp = mp_1 \circ mp_2$ ). However, the process itself, under the motivation of hearing the music as an outcome and the availability of the tool bars, led the student, who did not have a sophisticated background, to create the function according to the tutorial. Finally, to actually implement the denotator and to play the bass line in Rubato Composer, a network using rubettes (Figure 13) must be created. The student set up three rubettes; the Source rubette (Figures 10 and 11), the @AddressEval rubette (Figure 12), and the *ScorePlay* rubette.

The @AddressEval rubette is very interesting, because it is here that the student really can appreciate the functorial set up. As the *Note* denotators are composed of morphisms (algebraic structure, which cannot be implemented) with domain ("address")  $\mathbb{Z}^{63}$ , "null addressed" denotators must be generated. This is so that the denotator *Note* can be evaluated at a set of denotators of form *Score* and type *Power*. Hence the reason that, theoretically, the architectural concept framework uses functors (presheaves) that go from the category of modules with diaffine morphisms, to the category of sets.

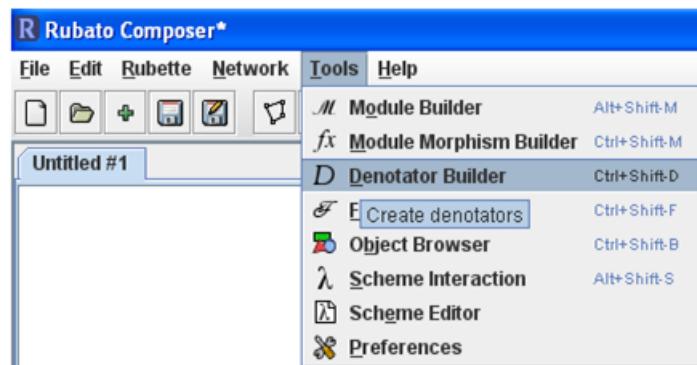


Figure 8: Denotator Builder.



Figure 9: The denotator Simplenote.

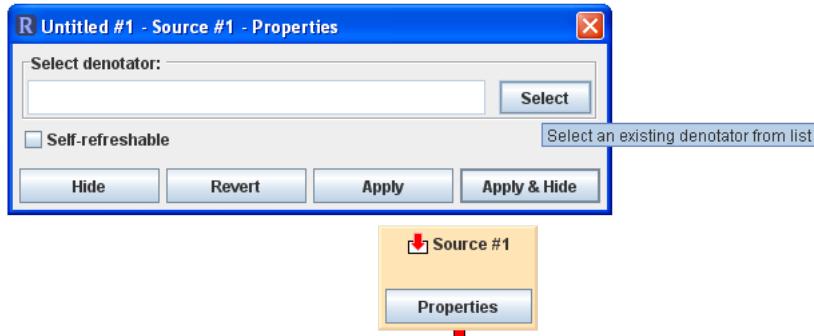
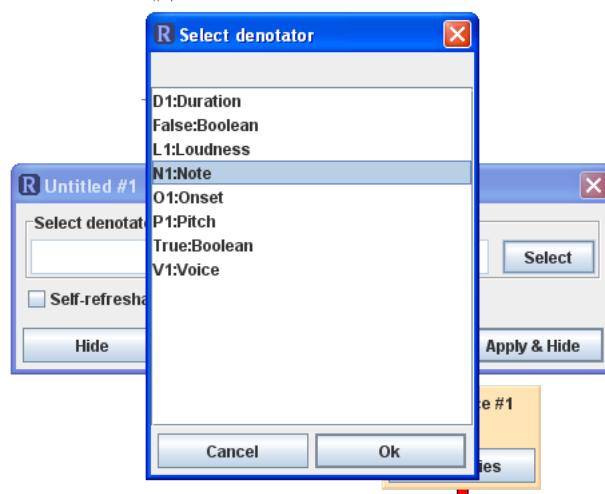


Figure 10: The Source Rubette.

Figure 11: Calling the *Note* denotator for the Source Rubette.

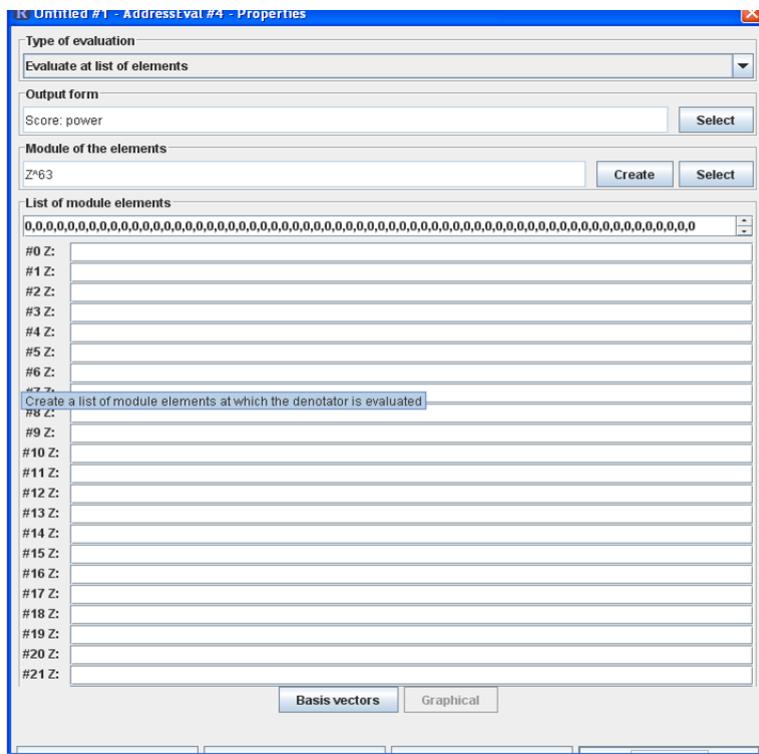


Figure 12: Output form for the @AdressEval rubette.



Figure 13: The network of rubettes.

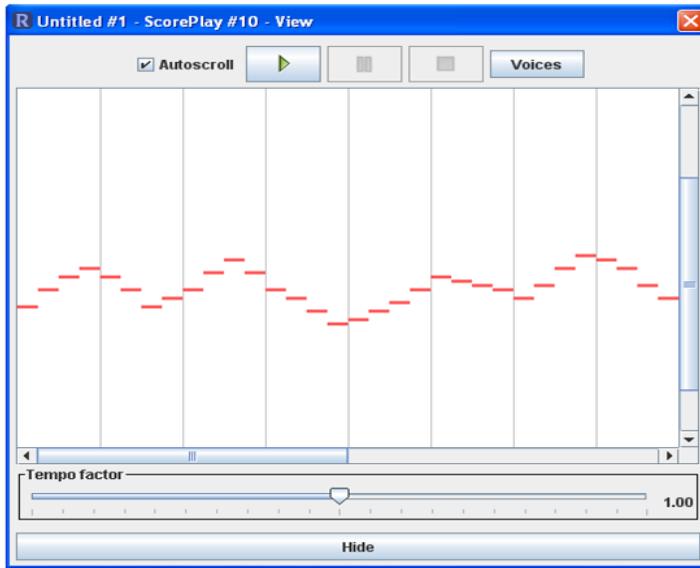


Figure 14: The implementation of the bassline of “All of Me”

The button *Basis Vectors* generates the canonical basis vectors, plus the zero vector (as we work with affine morphisms, which are linear transformations plus a translation). To play the bassline *ScorePlay* rubette is opened and connected to the *@AddressEval*. This is done the same way as the *Source* and *@AddressEval* rubettes.

Finally, once this low level implementation is carried out, by pushing the play button the final score (in this case the 64 note bassline) can be played via the Java software sequencer and synthesizer. It has a pianola format.

## 5 A Sample Unit

A sample unit has been created to show how the analysis and creation of a musical object in Rubato Composer can give students from different disciplines, in particular mathematics, computer science and music, a deeper understanding of abstract mathematics while satisfying aesthetic interests as well. It is also by these concrete musical objects that the abstract structures of mathematics, studied in group theory, modern and linear algebra in general, as well as geometry, set theory and category theory, are represented. For example, any regular musical pattern, such as a “groovy” rhythm, can represent an abstract categorical diagram. A musical transposition can be represented by an affine transformation, which can be seen as a translation, and the standard chromatic scale itself is isomorphic to  $\mathbb{Z}_{12}$ , with its “clock arithmetic”.

**Description of the Module:** The development of a melodic phrase, recursively transformed by transformations in the plane as ornamentation, using the *Wallpaper* rubette in RUBATO®Composer.

**Objectives and Activities:** All students (from all the disciplines) will be able to:

- Identify rigid transformations in the plane and give them musical meaning.  
For example:
  - mathematical translations – musical transpositions;
  - mathematical reflection – musical inversion, retrograde;
  - mathematical rotation – musical inversion-retrograde;
  - mathematical dilatation – musical augmentation in time;
  - mathematical shearing – musical arpeggios in time.
- All students will be able to:
  1. Use the software RUBATO®Composer and, in particular, the *Wallpaper* rubette, to generate musical ornamentation by means of diagrams of morphisms (functions).
  2. Create and interpret transformations, and compositions of transformations, like the following, in which  $f_1$  is a rotation of  $180^\circ$  followed by a translation, and  $f_2$  is a translation (Figure 15).
  3. Select any of the coordinates of a *Note* denotator (which is 5-dimensional) and combine them. When two coordinates are chosen, say *Onset* and *Pitch*, students will relate them to the rigid transformations in the Euclidean plane. Mathematics students (and those from computer science and other related areas) will formally construct the morphisms, while Music students can use the succession of primitive transformations by dragging with the mouse (Figure 16).
- Mathematics students will be able to construct the composition of module morphisms from the form *Note* to the form *Note*. For example, using the coordinates *Onset* and *Pitch*, they can construct the following composition

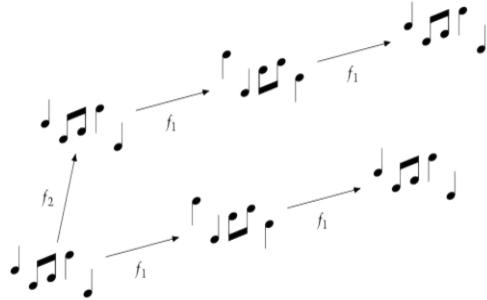


Figure 15: Example of affine transformations.

of embeddings, projections and affine transformations.

$$\begin{aligned} o_n &= p_1 \circ f \circ ((i_1 \circ o) + (i_2 \circ e_2 \circ p)) : A \rightarrow \mathbb{R}, \\ p_n &= c \circ p_2 \circ f \circ ((i_1 \circ o) + (i_2 \circ e_2 \circ p)) : A \rightarrow \mathbb{Q}. \end{aligned}$$

In order to combine these two morphisms into a single instance of  $\mathbb{R}^2$  they must be summed, so that onset and pitch become respective axes in  $\mathbb{R}$ . Then the transformation  $f$  is applied, and finally, to return the coordinates to the module morphisms  $o$  and  $p$ , the projections  $p_1$  and  $p_2$  are applied, where  $c$  represents  $\mathbb{R}$  quantized to  $\mathbb{Q}$ .

The *Wallpaper* rubette is an example of a low-level process where the work is done a very mathematical context. For the music student, for example, this will not always be appropriate, as mathematics may be a means rather than an end. A high level ‘extreme’ example is the *BigBang* rubette, developed by Thalmann and Mazzola [18] in the context of Mathematical Gesture Theory and Computer Semiotics. It is based on a general framework for geometric composition techniques. The student can do and hear musical composition and relations by dragging the mouse; however, even with the high level rubettes that are, and will be, available, it is possible to retrace the steps and uncover the mathematics behind their construction.

## 6 Conclusions

This work was written as an overview of how the software RUBATO® Composer, developed within the category-theoretic framework of modern mathematical music theory, can be used as a learning tool in interdisciplinary courses and research projects involving mathematics, music and computer science students. These projects can include the creation of learning materials and strategies for implementation. The approach is interdisciplinary, and draws upon the need to adapt the educational endeavor in general to the exigencies of modern theories in mathematics, humanities and computer science. The objective is to create

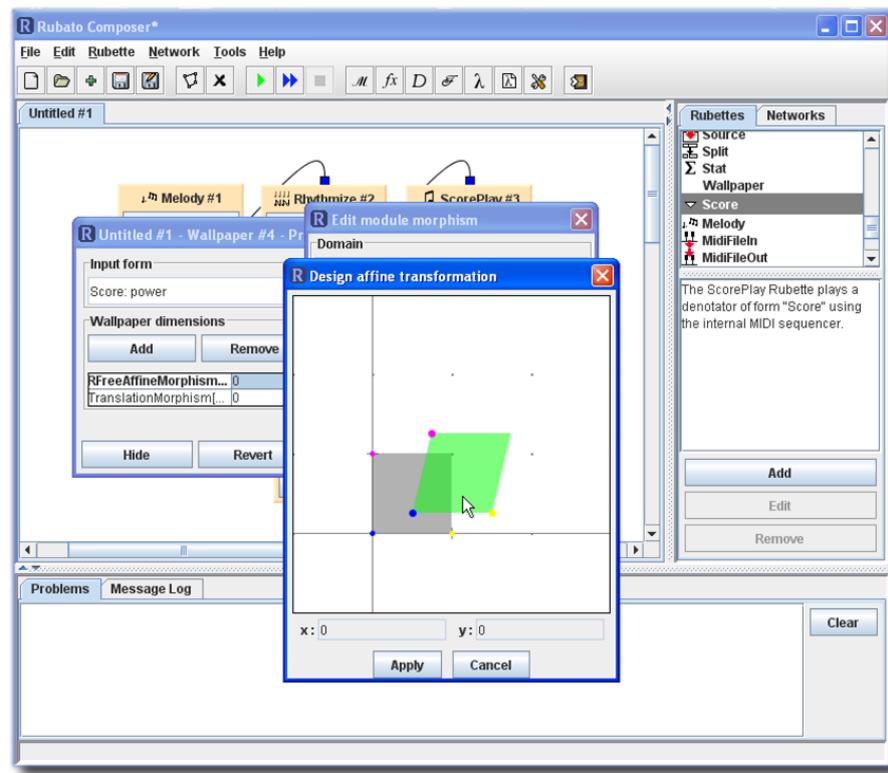
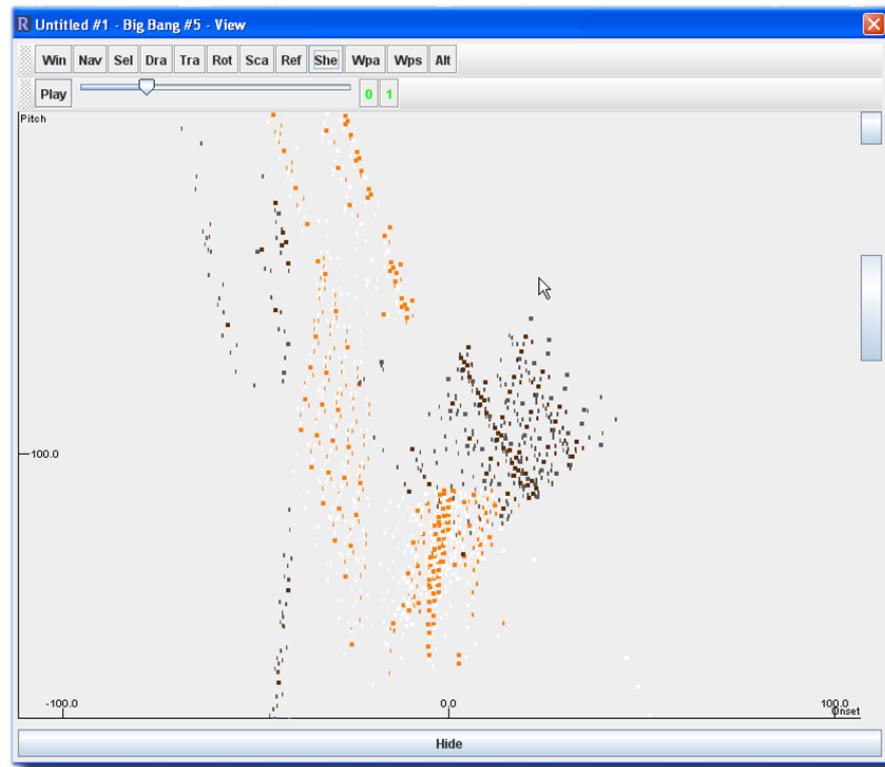


Figure 16: An affine transformation by dragging.

Figure 17: The *BigBang* rubette in action.

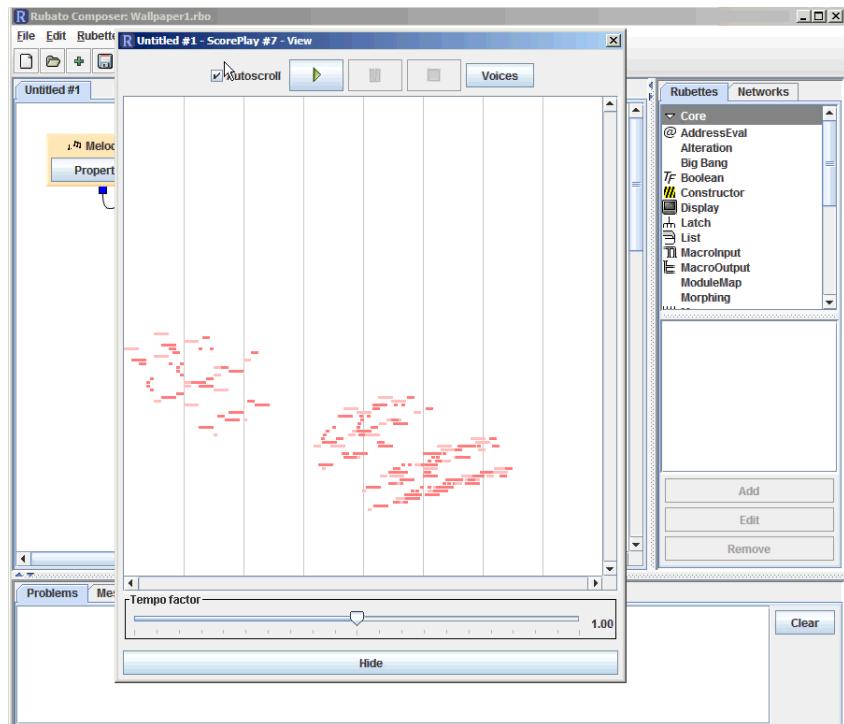


Figure 18: The pianola representation of a “composition” with the Big-Bang rubette.

materials and courses that can make abstract structures accessible, by representations using tangible, concrete musical objects. These types of materials and courses can introduce students into fields of mathematics that are usually reserved for those that have completed multiple semesters of prerequisites. Abstract algebra, linear algebra, set theory, category theory, and so on, are basic components of physics, computer science, logic, geometry, economics and even music theory. For this reason, there exists a strong need to make these areas accessible to a broad range of undergraduate students. The search for ways of enhancing advanced mathematical thinking has led to, among other methods, the use of the Rubato Composer system as a tool in the expression of these abstract concepts.

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# Some extraspecial (and almost extraspecial) musical transformation groups

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## Abstract

We examine finite groups (and subgroups) of musical transformations that are isomorphic to various extraspecial and almost extraspecial groups. These groups incorporate various 4-cycles that correspond to the imaginary unit  $i$  (i.e., a square root of  $-1$ ), and which intersect in a central involution. These transformation groups act on sets of pitch-classes in an octatonic collection, and in various triadic-transformational systems. We apply our results to an analysis of a large-scale inversional gesture in the musical composition *Cantata Profana* by Béla Bartók.

**Keywords:** transformation theory, neo-Riemannian theory, Lewin, octatonism, uniform triadic transformations, complex numbers, quaternions, group theory, quaternion group, Pauli group, extraspecial group, almost extraspecial group, central product, Bartók.

## 1 Introduction

In his 2009 *Journal of Mathematics and Music* article [1], Guerino Mazzola provides a gestural interpretation of the operation of inversion. This gesture conforms to a half rotation in the complex plane: “the resolution of the negation  $x \mapsto -x$  by a rotation in a new dimension is no overinterpretation of complex numbers. A strong argument for this ‘gestural’ reinterpretation of negation is in fact provided by the proof of the fundamental theorem of algebra using fundamental groups in the Gaussian plane.” We observe such a gesture in Figure 1. Here, the unit 1 moves in an arc through the imaginary unit  $i$  on the way to its inverse,  $-1$ . In this way,  $i$  functions as a square root of  $-1$  (as does  $-i$ ), and forms the basis of an imaginary dimension. Then, as we may describe other congruent gestures of order 4, all of which share the same square of order 2, we recognize the existence of additional imaginary units in further dimensions.

Mazzola’s inversional gesture is, by definition, continuous. In an effort to understand more fully how such gestures function in traditional music-analytical contexts, we extend his concept to discrete models. That is, we examine groups of musical transformations in a Lewinian sense [2] that are generated by these

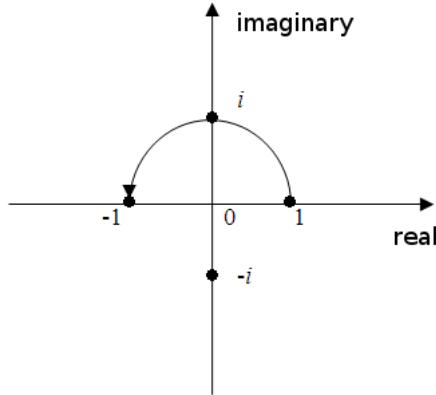


Figure 1: Inversion as a half rotation in the complex plane (after Mazzola 2009).

“imaginary transformations” [3], or which contain subgroups that are so generated. In particular, we describe isomorphisms from such these transformation groups to certain classes of extraspecial and almost extraspecial groups and subgroups, including quaternion, Pauli, dicyclic, and other related group structures. In the first section of this paper, we examine the actions and orbits of these groups on familiar collections of musical objects: pitch classes (and pitch-class sets) in octatonic space, and consonant triads in chromatic space. (Many of the results in this section derive from [3].) In the second section, we investigate imaginary transformations that act on the group of musical transposition operators themselves.

## 2 Extraspecial (sub)groups in octatonic pitch-class space

Our first example obtains in an octatonic system; hence, let  $O$  be the octatonic collection  $C, C\#, D\#, E, F\#, G, A, A\#$ , which we model here as a subset of the integers modulo 12, where  $C = 0$  in modular pitch-class space. One way to view  $O$  is as the union of two fully diminished seventh chords, each generated by minor third. Therefore, let us first consider the order-4 operation  $T_3$ , which represents musical transposition (mathematical translation) on  $O$  by  $+3 \bmod 12$ . In Appendix B of [2], David Lewin describes a Schritt-like contextual operation  $S_3$  of order 4 on  $O$  that commutes with  $T_3$  (Lewin actually calls this operation “Q3,” but we will be using the letter “Q” later for another structure). We observe further that  $S_3$  shares the same square as  $T_3$ . Then,  $S_3$  generates a cyclic group of order 4, and together  $T_3$  and  $S_3$  generate an abelian group of order 8, isomorphic to  $C_4 \times C_2$ . (Table 1 lists by label and in cyclic notation all the operations on  $O$  that appear throughout this section of the paper, including  $T_3$  and  $S_3$ .)

$T_0$	(0)(1)(3)(4)(6)(7)(9)(10)
$T_3$	(0, 3, 6, 9)(1, 4, 7, 10)
$T_3^{-1}$	(0, 9, 6, 3)(1, 10, 7, 4)
$S_3$	(0, 3, 6, 9)(1, 10, 7, 4)
$S_3^{-1}$	(0, 9, 6, 3)(1, 4, 7, 10)
$M_1$	(0, 1, 6, 7)(3, 4, 9, 10)
$M_1^{-1}$	(0, 7, 6, 1)(3, 10, 9, 4)
$N_1$	(0, 1, 6, 7)(3, 10, 9, 4)
$N_1^{-1}$	(0, 7, 6, 1)(3, 4, 9, 10)
$X_4$	(0, 4, 6, 10)(1, 3, 7, 9)
$X_4^{-1}$	(0, 10, 6, 4)(1, 9, 7, 3)
$Y_4$	(0, 4, 6, 10)(1, 9, 7, 3)
$Y_4^{-1}$	(0, 10, 6, 4)(1, 3, 7, 9)
$T_6 = (T_3^{\pm 1})^2 = (S_3^{\pm 1})^2 = (M_1^{\pm 1})^2$ $= (N_1^{\pm 1})^2 = (X_4^{\pm 1})^2 = (Y_4^{\pm 1})^2$	(0, 6)(1, 7)(3, 9)(4, 10)
$W_1$	(0, 7)(1, 6)(3, 10)(4, 9)
$W_2$	(0, 4)(1, 9)(3, 7)(6, 10)
$W_3$	(0, 6)(3, 9)
$I_1$	(0, 1)(3, 10)(4, 9)(6, 7)
$I_4$	(0, 4)(1, 3)(6, 10)(7, 9)
$I_7$	(0, 7)(1, 6)(3, 4)(9, 10)
$I_{10}$	(0, 10)(1, 9)(3, 7)(4, 6)

Table 1: Operations on  $O = \{0, 1, 3, 4, 6, 7, 9, 10\}$ .

A number of order-4 operations on  $O$  exist that have the same properties as  $S_3$ ; that is, they commute with  $T_3$  under multiplication, they share the same square as  $T_3$ , and together with  $T_3$  they generate a group isomorphic to  $C_4 \times C_2$ . One such operation is the inverse of  $S_3$ . Another pair of operations consists of the affine transformation we will call  $M_1$  and its inverse. Still another pair includes the product of  $S_3$  and  $M_1$ :  $X_4$ , and its inverse. Any two of the groups generated by  $T_3$  and any one of these 4-cycles intersect in the cyclic group generated by  $T_3$ . Furthermore, all eight 4-cycles that are found in these three groups share a mutual square.

## 2.1 An order-8 extraspecial group, $H$

$S_3$ ,  $M_1$ , and  $X_4$  generate together an order-8 group  $H$  that is isomorphic to the quaternion group  $Q_8$ .  $Q_8$  is a nonabelian group, consisting of an identity element 1, three elements of order 4:  $i$ ,  $j$ , and  $k$  (each imaginary unit that is, square roots of  $-1$ ), their inverses (also order-4 square roots of  $-1$ ), and their mutual square,  $-1$  (of order 2). Moreover, just as  $i$  forms the basis of an imaginary dimension, perpendicular to the real axis in Figure 1, we observe that  $i$ ,  $j$ , and  $k$  are all orthogonal to one another, and serve as bases of three imaginary dimensions. The quaternion group is an example of an extraspecial group. First, it is a  $p$ -group—its order, 8, is a power of a prime number, in this case 23. Second, its center—consisting of those elements in the group that commute with every element—is cyclic of order  $p$ . In this case, it is of order 2, the subgroup generated by  $-1$ . Third, its center is equal to its Frattini subgroup, where the Frattini subgroup of a group contains the intersection of all its proper maximal subgroups. Finally, the group modulo its center is an elementary abelian  $p$ -group—where an elementary abelian  $p$ -group is a finite abelian group, such that every nontrivial element has order  $p$ , and  $p$  is a prime. In this case,  $Q_8$  modulo its center is isomorphic to the Klein four group, in which every non-identity element has order 2. The mutual square of  $S_3$ ,  $M_1$ , and  $X_4$  is  $T_6 = (0, 6)(1, 7)(3, 9)(4, 10)$ . Then, just as we may give the following presentation for  $Q_8$ ,

$$Q_8 := \langle i, j, k | i^2 = j^2 = k^2 = ijk \rangle,$$

we may also give a corresponding presentation for  $H$ .

$$H := \langle S_3, M_1, X_4 | S_3^2 = M_1^2 = X_4^2 = S_3 M_1 X_4 \rangle$$

The quaternions, as a number system, form a four-dimensional extension of the two-dimensional complex numbers. Given real numbers  $a$  and  $b$ , and an imaginary unit  $i$ , we call  $a + bi$  a complex number. Now, let  $a$ ,  $b$ ,  $c$ , and  $d$  be real numbers, and define imaginary units  $i$ ,  $j$ , and  $k$  as above. We say that  $a + bi + cj + dk$  is a quaternion. Whereas the imaginary unit in the complex numbers lies on an axis perpendicular to the real axis, the imaginary units in the quaternion system lie on three separate axes, each at a right angle to the real axis and to one another. Therefore, just as continuous motion through the

powers of  $i$  describes a circumference on the surface of a unit 1-sphere (i.e., on a circle with radius 1, as in Figure 1), motions through the powers of  $i$ ,  $j$  and  $k$  describe mutually perpendicular circumferences on the surface of a unit 3-sphere (a four-dimensional sphere). Call  $H'$  the centralizer of  $H$  in the symmetric group on  $O$ , consisting of all operations on  $O$  that commute with every element of  $H$ . Because the action of  $H$  on  $O$  is simply transitive (or regular),  $H$  is isomorphic to  $H'$ . Then, as  $H$  is nonabelian,  $H \neq H'$ . Fiore and Satyendra 2005 [4] describe a centralizer such as  $H'$  as the dual group of  $H$ . In this case, just as  $H$  can be generated by  $S_3$ —a contextual transposition that reverses one of the four-cycles of  $T_3$ —the affine operation  $M_1$ , and their product  $X_4$ ; we may generate  $H'$  from the usual transposition  $T_3$ , a contextual affine transformation  $N_1$  that reverses one of the 4-cycles of  $M_1$ , and their product,  $Y_4$ , the contextual image of  $X_4$  (again, Table 1 lists all these various operations). We note that  $H \cap H' = Z(H) = Z(H') = \langle T_6 \rangle$ , a point that will be of further significance later.

## 2.2 An order-16 almost extraspecial group

Regaining  $T_3$  and adjoining it to  $H$  yields a group  $P_{T_3}$ , isomorphic to the single-qubit Pauli group  $P_1$  of order 16, which is used in quantum information science [5].  $P_1$  is comprised of tensor products of the Pauli matrices and the unit matrix. In quantum mechanics, the Pauli matrices are used as a mathematical representation of spin for elementary (spin- $\frac{1}{2}$ ) particles in three spatial dimensions, hence our connection to Mazzola's half rotation in the complex plane, extended to the three imaginary dimensions of the quaternions. We may generate by using three *Wechsel*-like transformations  $W_n$ , as follows.

$$\begin{aligned} W_1 &= (0, 7)(1, 6)(3, 10)(4, 9) \\ W_2 &= (0, 4)(1, 9)(3, 7)(6, 10) \\ W_3 &= (0, 6)(3, 9) \end{aligned}$$

contains  $H$  above as a normal subgroup. Therefore, in terms of products of the various  $W_n$  operations that generate , the generators of  $H$  are shown below.

$$W_1 W_2 = S_3, W_3 W_1 = M_1, W_2 W_3 = X_4.$$

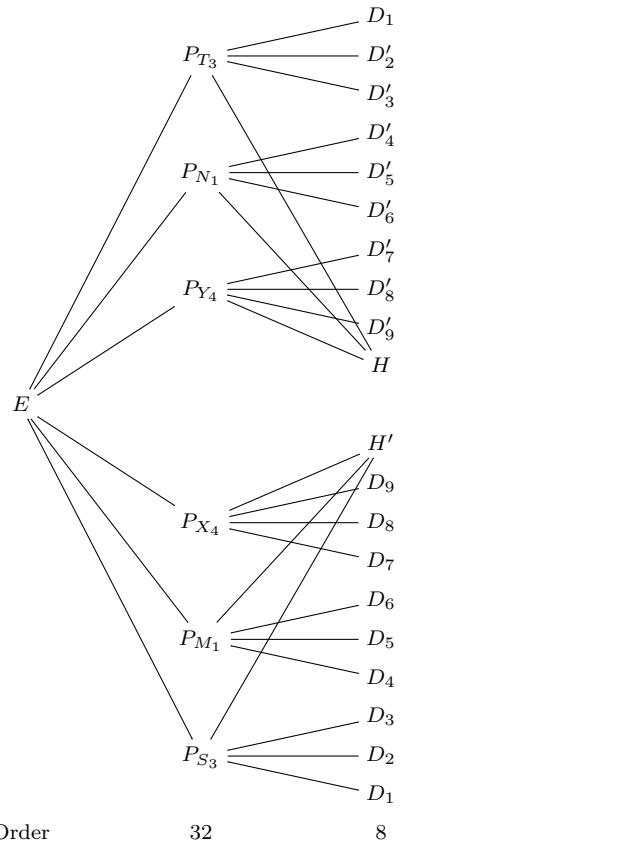
The center of of is generated by  $T_3$ . Another way to characterize is as an almost extraspecial group [6]. Like  $H$ , it is a  $p$ -group for  $p = 2$ , and its Frattini subgroup (generated by  $T_6$ ) has order  $p = 2$ . However, unlike an extraspecial group, its center is not equal to the Frattini subgroup; rather, it is of order 4, or  $p^2$ . As an almost extraspecial group, we may give as a central product [7], a type of structure that will become increasingly significant in our later discussion of larger extraspecial groups, to which it applies as well. Let  $G$  be a group, and let  $G_i$  and  $G_j$  be normal subgroups of  $G$ . Then, we may say that  $G$  is the central product  $G_i * G_j$  if  $G$  is generated by the elements of  $G_i$  and  $G_j$ , their commutator subgroup is trivial, and if  $G_i \cap G_j \leq Z(G)$ . For instance, we may

say that is a central product of  $H$  and the group generated by  $T_3$ , the members of which generate  $P_{T_3}$ . In this connection, we note that both  $H$  and the group generated by  $T_3$  are normal subgroups. Moreover, their commutator subgroup is trivial, and their intersection is a subgroup of the center of  $P_{T_3}$ .

### 2.3 The plus-type extraspecial group of order 32

If we adjoin to an inversion operator  $I_k$ , where  $k$  is 1, 4, 7, or 10 (i.e.,  $I_k$  consists of the product of all pairwise disjoint transpositions  $(a, b)$  in  $O$ , such that  $a + b = k \pmod{12}$ ), we obtain a group  $E$  with an action on  $O$  that is isomorphic to the plus-type extraspecial group of order 32. We observe that  $E$  can be generated by elements of  $H$  and  $H'$ ,  $H$  and  $H'$  are normal in  $E$ , and the commutator subgroup for  $H$  and  $H'$  is trivial. As we noted previously,  $H$  and  $H'$  intersect in the center of  $E$ , the group generated by  $T_6$ . We may therefore describe  $E$  as the central product of  $n$  copies of its extraspecial quaternion subgroups, in this case, as  $H * H'$  (for  $n = 2$ ). We could also give  $E$  as a central product of order-8 dihedral subgroups (the dihedral group of order 8's also being an example of an extraspecial group).  $E$  contains eighteen such subgroups, which form nine dually related pairs. One such pair consists of a subgroup  $D_1$  generated by transposition and inversion operators  $T_3$  and  $I_1$ , and a subgroup  $D'_1$  generated by contextual, *Schritt-* and *Wechsel*-like operators  $S_3, W_1$ . Again, we note  $D_1$  and  $D'_1$  are both normal in  $E$ ; their commutator subgroup is trivial, and their intersection is the center of  $E$ , the group generated by  $T_6$ . We recall that the generators of the order-16 group  $P_{T_3}$  above— $T_3, S_3, M_1$ , and  $X_4$ —all commute with  $T_3$  and are contained in  $E$ . We may accordingly give an alternate characterization of  $P_{T_3}$  in terms of  $E$ . Specifically,  $P_{T_3}$  is the centralizer in  $E$  of the group generated by  $T_3$ .  $E$  contains six such order-16 subgroups, each isomorphic to the single-qubit Pauli group  $P_1$  (hence, to the almost extraspecial group of order 16), and each centralizing one of the various 4-cycles we have defined above:  $S_3, M_1, X_4, T_3, N_1$ , and  $Y_4$  (and their respective inverses). Therefore, let us label these subgroups  $P_g$ , where  $g$  is one of those six operations. The  $P_g$  subgroups for which  $g$  is  $T_3, N_1$ , or  $Y_4$  all contain  $H$  as a subgroup. Those for which  $g$  is  $S_3, M_1$ , or  $X_4$  each contain  $H'$ . Figure 2 presents a lattice of all these various subgroups of  $E$ .

So far, we have examined the actions of various extraspecial groups on pitch-classes in an octatonic collection. We might also consider their actions on pitch-class sets. For instance, the orbit of the pitch-class set  $C^+ = \{0, 4, 7\}$  under  $E$  is of order 32. It consists of eight members each of four set classes:  $[0, 3, 7]$ ,  $[0, 1, 4]$ ,  $[0, 1, 3]$ , and  $[0, 2, 5]$  (i.e., all the trichordal SCs in  $O$  whose members do not contain a tritone). Furthermore, given the simply transitive action of  $E$  on the orbit of  $C$  under  $E$ ,  $C^E$ , it suggests the incorporation of a Generalized Interval System [2] that would accommodate all these sets of different types in an octatonic context.



$$\begin{aligned}
 D_1 &= \langle T_3, I_1 \rangle & D_4 &= \langle T_3, I_1 X_4 \rangle & D_7 &= \langle T_3, I_1 M_1 \rangle & D'_7 &= \langle S_3, N_1 I_7 \rangle & D'_4 &= \langle S_3, W_1 Y_4 \rangle & D'_1 &= \langle S_3, W_1 \rangle \\
 D_2 &= \langle N_1, I_7 \rangle & D_5 &= \langle N_1, I_7 X_4 \rangle & D_8 &= \langle N_1, I_7 M_1 \rangle & D'_8 &= \langle M_1, W_3 N_1 \rangle & D'_5 &= \langle M_1, W_3 \rangle & D'_2 &= \langle M_1, W_3 \rangle \\
 D_3 &= \langle Y_4, I_{10} \rangle & D_6 &= \langle Y_4, I_{10} X_4 \rangle & D_9 &= \langle Y_4, I_{10} M_1 \rangle & D'_9 &= \langle X_4, W_2 N_1 \rangle & D'_6 &= \langle X_4, W_2 Y_4 \rangle & D'_3 &= \langle X_4, W_2 \rangle
 \end{aligned}$$

Figure 2: Lattice of order-16 almost extraspecial and order-8 extraspecial subgroups of \$E\$.

### 3 Extraspecial subgroups in triadic transformational systems

Thus far, we have described the actions of various extraspecial and almost extraspecial groups on octatonic system, building increasingly larger groups of transformations that incorporated them. In this section, we will investigate their functioning in triadic transformational systems: specifically, in Julian Hook's 2002 [8] uniform triadic transformations (UTTs) and quasi-uniform triadic transformations (QTTs). Whereas these transformational systems are defined in terms of group actions on the entire order-24 set-class  $K$  of consonant major and minor triads, as Hook points out, they may also be extended easily to other sets: transpositionally and inversionally asymmetrical set-classes, pairs of inversionally symmetric set-classes, and so on. For now, however, we will limit our discussion to triads.

#### 3.1 Quaternion subgroups in the UTTs

We begin with Hook's group of UTTs, which we label  $U_1$ . We give a UTT  $u \in U_1$  using the notation:  $u = \langle \sigma, t^+, t^- \rangle$ , where  $t^+ \in \mathbb{Z}_{12}$  is the index of (musical) transposition on the set of major triads,  $t^- \in \mathbb{Z}_{12}$  is the index of transposition on the set of minor triads, and  $\sigma$  is either  $+$ , which preserves modes, or  $-$ , which reverses them (while preserving triadic roots). We observe that  $U_1$  is a wreath product of order  $122^2 \cdot 2 = 288$ . We find three conjugate quaternion subgroups  $H_i$  in  $U_1$ , shown here in Table 2. Then, we note that the intersection of these three subgroups is the subgroup generated by  $\langle +, 3, 9 \rangle$ , an order-4 *Schritt*.

The smallest subgroup of  $U_1$  that contains all three  $H_i$ ,  $R_1^*$ , is of order 24. Hook's set of skew-Riemannian UTTs, which consists of twelve *Schritts* and twelve skew-*Wechsels* (where a skew-*Wechsel* is a mode-reversing operation in which the  $t^+$  and  $t^-$  indices of sum to  $6 \bmod 12$ ).  $R_1^*$  is isomorphic to the dicyclic group  $\text{Dic}_n$  for  $n = 6$ . We may view  $\text{Dic}_n$  as a subgroup of the quaternions as a number system, wherein the subgroup is generated by an element  $a$  of order  $2n$ , where  $i$  is an imaginary unit; and an element  $x$  of order 4, where  $x = j$ , an imaginary unit orthogonal to  $i$ . In the specific case of  $R_1^*$ , the group can be generated by a mode-preserving element  $S_1$  of order 12 (a unit *Schritt*) and a mode-reversing element  $X$  of order 4 (a skew-*Wechsel*). Then, one possible presentation of  $R_1^*$  appears as follows.

$$R_1^* = \langle S_1, X | S_1^{12} = 1, X_2 = S_1^6, X^{-1}S_1X = S_1^{-1} \rangle.$$

We note further that  $R_1^*$  has a simply transitive action on  $K$ , thus describing one and only one motion from any one triad to any other.

#### 3.2 Additional quaternion subgroups in the QTTs

Hook's QTTs obtain by adjoining the usual inversion operation to the UTTs. They form a wreath product of order 1152, which we label  $Q$ . Slightly modified

$H_1 = \langle\langle +, 3, 9 \rangle, \langle -, 0, 6 \rangle, \langle -, 3, 3 \rangle \rangle$
$\langle +, 3, 9 \rangle = (C^+, Eb^+, F\#^+, A^+)(C\#^+, E^+, G^+, Bb^+)(D^+, F^+, Ab^+, B^+)$
$(C\#^-, Bb^-, G^-, E^-)(D^-, B^-, Ab^-, F^-)(Eb^-, C^-, A^-, F\#^-)$
$\langle -, 0, 6 \rangle = (C+, C^-, F\#^+, F\#^-)(C\#^+, C\#^-, G^+, G^-)(D^+, D^-, Ab^+, Ab^-)$
$(Eb^+, Eb^-, A^+, A^-)(E^+, E^-, Bb^+, Bb^-)(F^+, F^-, B^+, B^-)$
$\langle -, 3, 3 \rangle = (C+, Eb^-, F\#^+, A^-)(C\#^+, E^-, G^+, Bb^-)(D^+, F^-, Ab^+, B^-)$
$(Eb^+, F\#^-, A^+, C^-)(E^+, G^-, Bb^+, C\#^-)(F^+, Ab^-, B^+, D^-)$
$H_2 = \langle\langle +, 3, 9 \rangle, \langle -, 1, 5 \rangle, \langle -, 4, 2 \rangle \rangle$
$\langle +, 3, 9 \rangle = (C+, Eb^+, F\#^+, A^+)(C\#^+, E^+, G^+, Bb^+)(D^+, F^+, Ab^+, B^+)$
$(C\#^-, Bb^-, G^-, E^-)(D^-, B^-, Ab^-, F^-)(Eb^-, C^-, A^-, F\#^-)$
$\langle -, 1, 5 \rangle = (C+, C\#^-, F\#^+, G^-)(C\#^+, D^-, G^+, Ab^-)(D^+, Eb^-, Ab^+, A^-)$
$(Eb^+, E^-, A^+, Bb^-)(E^+, F^-, Bb^+, B^-)(F^+, F\#^-, B^+, C^-)$
$\langle -, 4, 2 \rangle = (C+, E^-, F\#^+, Bb^-)(C\#^+, F^-, G^+, B^-)(D^+, F\#^-, Ab^+, C^-)$
$(Eb^+, G^-, A^+, C\#^-)(E^+, Ab^-, Bb^+, D^-)(F^+, A^-, B^+, Eb^-)$
$H_3 = \langle\langle +, 3, 9 \rangle, \langle -, 2, 4 \rangle, \langle -, 5, 1 \rangle \rangle$
$\langle +, 3, 9 \rangle = (C+, Eb^+, F\#^+, A^+)(C\#^+, E^+, G^+, Bb^+)(D^+, F^+, Ab^+, B^+)$
$(C\#^-, Bb^-, G^-, E^-)(D^-, B^-, Ab^-, F^-)(Eb^-, C^-, A^-, F\#^-)$
$\langle -, 2, 4 \rangle = (C+, D^-, F\#^+, Ab^-)(C\#^+, Eb^-, G^+, A^-)(D^+, E^-, Ab^+, Bb^-)$
$(Eb^+, F^-, A^+, B^-)(E^+, F\#^-, Bb^+, C^-)(F^+, G^-, B^+, C\#^-)$
$\langle -, 5, 1 \rangle = (C+, F^-, F\#^+, B^-)(C\#^+, F\#^-, G^+, C^-)(D^+, G^-, Ab^+, C\#^-)$
$(Eb^+, Ab^-, A^+, D^-)(E^+, A^-, Bb^+, Eb^-)(F^+, Bb^-, B^+, E^-)$

Table 2: Quaternion subgroups in the UTTs.

from the notation for the UTTs, we give a QTT  $u$  as  $\langle \sigma, t^+, t^-, \rho^+, \rho^- \rangle$ . Again,  $t^+$  and  $t^-$  are the indices of transposition for major and minor triads, and  $\sigma$  is either  $+$  for mode-preserving or  $-$  for mode-reversing; but, before applying  $\sigma$ , we need to consider  $\rho^+$  and  $\rho^-$ . If  $\rho^+$  is  $+$ , then no further action is needed on the major triads. If it is  $-$ , however, exchange each major triad with the one whose root is the pitch-class inverse the root of the original. The same holds for the minor triads and  $\rho^-$ . Then we apply  $\sigma$ . The UTTs thus form a subgroup of the QTTs, for which  $\rho^+$  and  $\rho^-$  are both equal to  $+$ .

Three additional quaternion subgroups  $H'_i$  appear in  $Q$  (Table 3), dual to those contained only in  $U$ . The intersection of these subgroups is a cyclic subgroup generated by an order-4 transposition, rather than a *Schritt*. As with the  $H_i$  in  $R_1^*$ , the  $H'_i$  are all contained in an order-24 dicyclic subgroup  $R_{11}^*$ , dual to  $R_1^*$ , which also has a simply transitive action on  $K$ . Then,  $R_{11}^*$  is a subgroup of an order-288 dual of  $U_1$ ,  $U_{11}$ . We may derive  $U_{11}$  from  $U_1$  (and  $R_{11}^*$  from  $R_1^*$ , etc.) and vice versa, under conjugation by  $F^+ = \langle +, 0, 0, -, + \rangle$ , one of Hook's simple "flips," which turns *Schritts* into transpositions, and skew-*Wechsels* into skew-inversions. The three  $H_i$  and the three  $H'_i$  produce nine plus-type extraespecial subgroups of order 32 in  $Q$ :  $E_i$ , where  $1 \leq i \leq 9$ . Each of these subgroups is a central product of one quaternion subgroup of  $R_1^*$  by one from  $R_{11}^*$ . Whereas all nine subgroups are conjugate to one another in  $Q$ , we note

$H'_1 = \langle\langle +, 3, 3, +, + \rangle, \langle -, 9, 3, -, - \rangle, \langle -, 0, 6, -, - \rangle \rangle$
$\langle +, 3, 3, +, + \rangle = (C^+, Eb^+, F\#^+, A^+)(C\#^+, E^+, G^+, Bb^+)(D^+, F^+, Ab^+, B^+)$
$(C\#^-, E^-, G^-, Bb^-)(D^-, F^-, Ab^-, B^-)(Eb^-, F\#^-, A^-, C^-)$
$\langle -, 9, 3, -, - \rangle = (C^+, Eb^-, F\#^+, A^-)(C\#^+, D^-, G^+, Ab^-)(D^+, C\#^-, Ab^+, G^-)$
$(Eb^+, C^-, A^+, F\#^-)(E^+, B^-, Bb^+, F^-)(F^+, Bb^-, B^+, E^-)$
$\langle -, 0, 6, -, - \rangle = (C^+, C^-, F\#^+, F\#^-)(C\#^+, B^-, G^+, F^-)(D^+, Bb^-, Ab^+, E^-)$
$(Eb^+, A^-, A^+, Eb^-)(E^+, Ab^-, Bb^+, D^-)(F^+, G^-, B^+, C\#^-)$
$H'_2 = \langle\langle +, 3, 3, +, + \rangle, \langle -, 8, 2, -, - \rangle, \langle -, 11, 5, -, - \rangle \rangle$
$\langle +, 3, 3, +, + \rangle = (C^+, Eb^+, F\#^+, A^+)(C\#^+, E^+, G^+, Bb^+)(D^+, F^+, Ab^+, B^+)$
$(C\#^-, E^-, G^-, Bb^-)(D^-, F^-, Ab^-, B^-)(Eb^-, F\#^-, A^-, C^-)$
$\langle -, 8, 2, -, - \rangle = (C^+, E^-, F\#^+, Bb^-)(C\#^+, Eb^-, G^+, A^-)(D^+, D^-, Ab^+, Ab^-)$
$(Eb^+, C\#^-, A^+, G^-)(E^+, C^-, Bb^+, F\#^-)(F^+, B^-, B^+, F^-)$
$\langle -, 11, 5, -, - \rangle = (C^+, C\#^-, F\#^+, G^-)(C\#^+, C^-, G^+, F\#^-)(D^+, B^-, Ab^+, F^-)$
$(Eb^+, Bb^-, A^+, E^-)(E^+, A^-, Bb^+, Eb^-)(F^+, Ab^-, B^+, D^-)$
$H'_3 = \langle\langle +, 3, 3, +, + \rangle, \langle -, 10, 4, -, - \rangle, \langle -, 1, 7, -, - \rangle \rangle$
$\langle +, 3, 3, +, + \rangle = (C^+, Eb^+, F\#^+, A^+)(C\#^+, E^+, G^+, Bb^+)(D^+, F^+, Ab^+, B^+)$
$(C\#^-, E^-, G^-, Bb^-)(D^-, F^-, Ab^-, B^-)(Eb^-, F\#^-, A^-, C^-)$
$\langle -, 10, 4, -, - \rangle = (C^+, D^-, F\#^+, Ab^-)(C\#^+, C\#^-, G^+, G^-)(D^+, C^-, Ab^+, F\#^-)$
$(Eb^+, B^-, A^+, F^-)(E^+, Bb^-, Bb^+, E^-)(F^+, A^-, B^+, Eb^-)$
$\langle -, 1, 7, -, - \rangle = (C^+, B^-, F\#^+, F^-)(C\#^+, Bb^-, G^+, E^-)(D^+, A^-, Ab^+, Eb^-)$
$(Eb^+, Ab^-, A^+, D^-)(E^+, G^-, Bb^+, C\#^-)(F^+, F\#^-, B^+, C^-)$

Table 3: Additional quaternion subgroups in the QTTs.

the following relations under conjugation by  $F^+$ .

$$H_1 * H'_3 = E_1 \xrightarrow{F^+} E_9 = H'_1 * H_3,$$

$$H_1 * H'_2 = E_2 \xrightarrow{F^+} E_8 = H'_1 * H_2,$$

$$H_2 * H'_3 = E_3 \xrightarrow{F^+} E_7 = H'_2 * H_3,$$

( $E_4$ ,  $E_5$ , and  $E_6$  are all self-conjugate under  $F^+$ .) Figure 3 shows a lattice of all these various subgroups of  $Q$ , and their dual relations under conjugation by  $F^+$ .

## 4 Imaginary transformations as hyper-operators

The preceding discussion presents various models of extra and almost extraespecial groups and subgroups with actions in pitch-class space, first on an octatonic collection, and then on the set of consonant triads. Nevertheless, such models have obvious limitations with regard to the concept of “imaginary transformations.” For instance, the squares of these transformations invert various other transformations that have order 4, such as  $T_3$ , but they do not invert multiples of those transformations, such as  $T_6$ ; nor do they invert a more basic unit, such as  $T_1$ . To do so, we might define a system in which such imaginary transformations are applied exponentially, rather than multiplicatively that is, transforming a unit under conjugation. In other words, we consider imaginary transformations that behave like hyper-operators that derive from inner auto-

Order

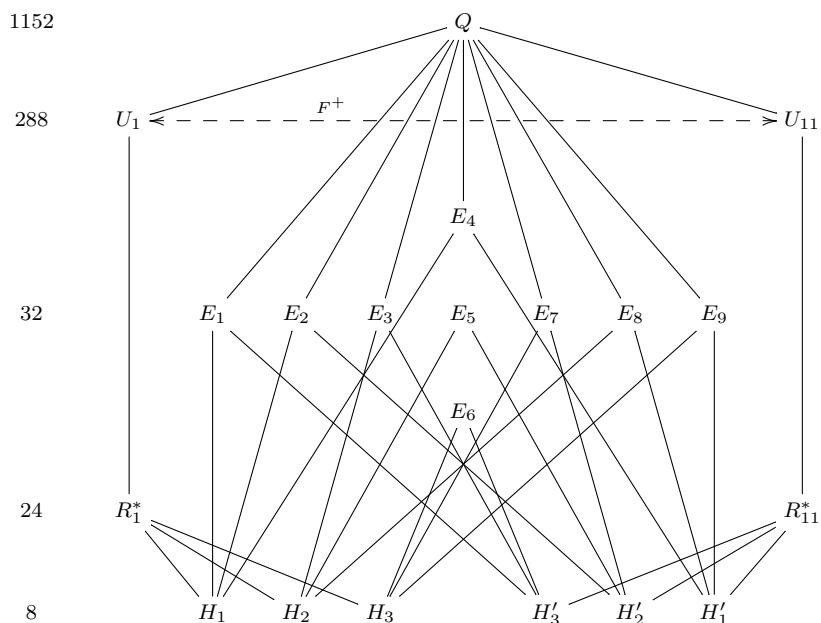


Figure 3: Lattice of subgroups of  $Q$ , with dual relations under conjugation by  $F^+$ .

morphisms in the sense of Klumpenhouwer [9]. Let  $T$  be the usual (mod 12) transposition group; we may describe  $T$  as a finite quotient of a discrete lattice in the real axis. Then, call an operation  $J$  an imaginary transformation if  $T_n$  conjugated by  $J^2$  equals the inverse transposition,  $T_{-n}$ , for all  $T_n$  in  $T$ . We return to Hook's quasi-uniform triadic transformations. We have already seen the simple flip  $F^+$ , which is non-uniform on the major triads. Hook defines another simple flip,  $F^-$ ,  $F^- = \langle +, 0, 0, +, - \rangle$ , which is non-uniform on the minor triads. Then, their product is the double flip  $F^+F^- = F^-F^+ = \langle +, 0, 0, -, - \rangle$ . Hook notes that these flips are not inversions, as they are mode-preserving. We note, however, an interesting and pertinent property of the double flip: it inverts  $T_n = \langle +, n, n, +, + \rangle$  under conjugation (for any  $n$ ). In fact, any such double flip  $V$  acts on  $T_n$  in this way. The reason has to do with the fact that the double flip is *in a sense* a type of inversion. It is simultaneously the inversion  $I_{-x}$  on the *roots* of major triads, and the inversion  $I_{-y}$  on the *roots* of the minors. As a result, it inverts  $T_n$  under conjugation, as does an inversion. Now we just need to determine the square root for any such double flip, and its effect on  $T_n$  under conjugation. As it turns out, there is not just one square root, but four square roots:

$$\begin{aligned} J^+ &= \langle -, p, q, +, - \rangle, p + q = x \quad \text{and} \quad q - p = y, \\ J_6^+ &= \langle -, p + 6, q + 6, +, - \rangle, p + q = x \quad \text{and} \quad q - p = y, \\ J^- &= \langle -, p, q, -, + \rangle, p - q = x \quad \text{and} \quad q + p = y, \\ J_6^- &= \langle -, p + 6, q + 6, -, + \rangle, p - q = x \quad \text{and} \quad q + p = y, \end{aligned}$$

two quasi-flips (quasi, because they are mode-reversing, not mode-preserving like the true simple flips) that are non-uniform on minor triads:  $\langle -, p, q, +, - \rangle$ , such that  $p + q = x$  and  $q - p = y$  (and, having chosen one suitable pair  $p$  and  $q$ , by virtue of  $T_6$ 's generating the center of the  $T/I$  group, the other pair is  $p+6$  and  $q+6$ ); and two quasi-flips that are non-uniform on major triads. Conjugating  $T_n$  by any such quasi-flip yields a *Schritt* of the same order as  $T_n$ : call the conjugate  $S_n$  for conjugation by quasi-flips that are non-uniform on minor triads, such as  $J^+$ ; and call it  $S_{-n}$  for those that are non-uniform on the major triads, including  $J^-$ . In other words, triadic transformational theory presents transposition as translation in two  $T$ -orbits *with the same orientation*. Conjugation by some  $J$ , then, inverts one (or the other) of those orbit's translations. Finally, conjugation by  $J$ -squared inverts both from their original orientation.

#### 4.1 *Cantata Profana*

A relevant analytical context may be found in Bartók's *Cantata Profana*. It tells a story (which Bartók translated from two Romanian colinde folk songs), common to several cultures of the Danube basin, of a father who sends his nine sons into the forest to hunt, where they are instead magically turned into deer. The work begins with a particular non-diatonic scale on D (Figure 4),

In fact, the non-diatonic scale at the beginning is the inverse of the scale at the end: call it an "undertone" scale on D. (A number of theorists have writ-



Figure 4: Undertone scale in Bartók's *Cantata Profana*, I, 1-5 and ends with an overtone (or "acoustic") scale, also on D (Figure 5).



Figure 5: Overtone scale in Bartók's *Cantata Profana*, III, 79-87.

Figure 6: Bartók, *Cantata Profana*, II, 1-4.

ten about the relationship between these scales in *Cantata Profana*, including Antokoltez [10] who views them not as inversions of one another, but rather as rotations. Indeed, they are both: much like the ionian and phrygian modes are simultaneously rotations and inversions of each other. We adopt the inversional viewpoint here for purposes of illustration of our premise.) The central transformation of sons into deer in the folk tale and in particular the attendant redefinition of their relationship with their father, i.e., “we can never go home” is thus represented musically in the inversion of the undertone scale to the overtone scale, where the latter could be said to signify the world of Nature of which they are now a part. But, what happens musically at the midpoint of this large-scale inversional gesture? According to the model we developed above, we would expect some sort of *Schritt*-like activity. In fact, at the beginning of the second movement at the point in the piece just following the sons’ transformation to deer, and just prior to their father’s discovery of his sons’ new physiologies we find just such a texture (Figure 6).

In a passage that relates to the music at the beginning of the first movement, and to that at the end of the last by virtue of its rhythmic constant, its orchestration in the strings, and to some extent its stepwise motion we find a passage that incorporates *Schritt*-like movement between two T-orbits: singleton pitch-classes on the one hand, and major thirds on the other. As Hook states, triadic-transformational theory is not limited only to triads, it can apply to other asymmetric set-classes, or to pairs of set-classes of inversionally symmetric pcsets, in this case to the union of set classes [0] and [0, 4]. We can therefore define *Schritt*-like UTTs that send members of these set-classes in opposite directions. Further regarding the stepwise connection of this passage to the scalar materials that border the piece: whereas the latter are primarily unidirectional in pitch space, the music in each of the respective T-orbits in this passage follows a more complex contour. They rise and fall in such a way that describes unfolded *Schritt*-like movements of trichordal cells: one rises by whole step and half step, then descends by the same intervals, and so on. (And, of course, the mirror image of this motion appears in the other orbit.) It is as if, here, at this magical moment in the piece, we find *Schritte* embedded within *Schritte*, where such transformations may be said to represent a square

Figure 7: Unfolded *Schritte* embedded within *Schritte*.

root of the large-scale inversional gesture that links the undertone scale at the beginning of the piece to the overtone scale at its end (Figure 7).

## 5 Conclusions

In summary, we have defined a number of musical transformation groups and subgroups that are isomorphic to various extraspecial and almost extraspecial groups. Such transformation groups incorporate “imaginary transformations.” However, such an “imaginary transformation” in a Lewinian-transformational sense that is fully isomorphic to the imaginary number  $i$  is not really possible—such transformations are truly *imaginary*—but a number of related situations yield interesting and useful music-theoretical notions. In the first part of this paper, we constructed various group structures that have quaternion subgroups, where the quaternion group itself is generated by three orthogonal imaginary units. What is really lacking in these structures is a corresponding image of the real axis. What precisely is the unit (and its multiples) that the squares of these “imaginary transformations” invert? Ironically, *that* is what is “imaginary” in this context. Nevertheless, our results led to interesting and potentially useful transformation groups. In the second part, we examined operations whose square inverts a unit transposition (and any multiple thereof) *under conjugation*. Of course, conjugation is not multiplication—it is essentially a form of exponentiation whereas the imaginary number  $i$  squared inverts the real axis under multiplication, not exponentiation. Rather, what we define here is more akin to a “logarithm to base  $T_1$ ,” which includes exponents that invert the base, and which themselves have various square roots. Nonetheless, such transformations may help us understand large-scale inversional gestures from a transformational perspective.

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# **La teoría *d1*, MúSIIC-Win y algunas aplicaciones al análisis musical: *Seis piezas para piano*, de Arnold Schoenberg**

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## **Resumen**

Este texto presenta conceptos básicos de la teoría *d1* sobre el potencial combinatorio de los intervalos de las escalas. La teoría *d1* no se asocia a un sistema de composición ni propone una estética para organizar los materiales musicales, sino que constituye una herramienta de búsqueda en creación, análisis, pedagogía, musicología y etnomusicología. Su más reciente desarrollo es el programa informático MúSIIC-Win, que de la posibilidad de abordar el potencial combinatorio de los intervalos en escalas de duración o de altura que comprendan de 3 a 24 términos. La teoría *d1* es particularmente útil en el análisis musical de partituras basadas en escalas modales, tonales, cromática, o de macro o micro-intervalos. La base de la teoría son las clases de intervalo, idea desde la cual se observan las agrupaciones secuenciales o verticales de intervalos, las identidades y sus permutaedros, que proveen información precisa sobre la combinatoria que conduce a su vez a entender la idea de la exploración musical de escalas de duración y/o de altura a las que recurre una obra. El artículo concluye con el análisis de las 6 piezas para piano de Schoenberg, composición que revela bajo distintos ángulos una exploración inédita de la tonalidad así como la inmersión gradual en la atonalidad.

## **1. Discontinuo y continuo**

Las posibilidades de estructurar elementos musicales como el sonido, el ritmo y el espacio de representación dependen de las características de su materia como de los límites de la percepción. Ello hace que la materia musical tienda a dividirse en dos grandes campos, el *discontinuo*, que incluye toda escala que pueda formarse con un número finito de términos, o el *continuo*, donde la percepción deja de distinguir un punto dado de aquellos que le son adyacentes —por ejemplo, en un *glissando* de altura. El discontinuo contrasta con la impresión unitaria que ofrece el continuo, de modo que la oposición de sus características se extiende al intento mismo de organizarlos:

- La menor resolución escalar en el discontinuo propicia las operaciones combinatorias con términos o intervalos. Al posibilitar un mayor control, la organización escalar predomina a lo largo de la historia de la música; ello no obstante, el caso de la altura sonora ofrece una persistente limitación de la combinatoria impuesta por sistemas musicales de composición más inspirados en la práctica que en criterios científicos para tratar los materiales de tipo discontinuo.
- La mayor cantidad de información en el continuo dificulta las operaciones de cálculo o la discriminación perceptiva de referencias estables, aun a pesar de su mayor sencillez estructural, que incita a suponer una manipulación dúctil y ajena al cálculo. Los ámbitos de tipo continuo tienden a integrarse más a los modos de ejecución —v.g. *glissandi*, *portamenti*, *rubati*, *accelerandi* u otros— que a una organización teórica. Dada la vastedad de la información en el continuo, su manipulación dentro de la escala resulta un exceso en la práctica musical [6].

La organización teórica de los materiales de orden discontinuo y continuo se apoya en dos nociones de base:

- resolución escalar: número de puntos existentes en una recta y diversidad cuantitativa de dimensiones escalares<sup>1</sup>,
- mínima distancia: medición de un punto cualquiera a sus adyacentes inmediato anterior y posterior y unidad de base para medir los intervalos de las escalas.

El estudio de la combinatoria de los intervalos de las escalas requiere considerar su resolución como el producto de una división entre un número finito de partes. Éstas pueden ser el producto de divisiones iguales o desiguales, si bien el ensayo de generalización pide para mayor facilidad recurrir a divisiones iguales, incluso si éstas pudiesen convertirse después a valores desiguales. La operación con divisiones iguales permite una manipulación con números enteros que facilita el cálculo de la combinatoria y sintetiza la información, un apoyo importante para los mecanismos mentales de cálculo, memoria y percepción que intervienen en la operación con los intervalos.

Si bien un mínimo de tres términos basta para constituir la escala más sencilla, fijar un número de términos en el límite superior es una cuestión de orden práctico relativo a cada tipo de material —escalas de altura o de duración— y que remite a las capacidades de cálculo, memoria o percepción de los materiales de tipo continuo<sup>2</sup>. En general, la expansión de la combinatoria de cada escala es el límite principal para operar en la práctica con la riqueza de sus materiales.

<sup>1</sup>La resolución escalar es algo menos significativa en el continuo y se entiende mejor bajo la noción de color de la resolución, idea que permite, estructural o perceptivamente, distinguir micro-divisiones del tono —por ejemplo diferenciar decimoquintos de decimosextos de tono.

<sup>2</sup>En los intervalos de altura se adopta desde el Renacimiento el límite perceptivo del noveno de tono, o coma, con base en las divisiones de la octava propuestas por el cartógrafo Gerhardus Mercator; véase [13].

## 2. Conceptos básicos

### 2.1. Ámbitos y escalas

#### 2.1.1. Ámbitos continuo y discontinuo

El continuo es un espacio lineal integrado por un conjunto ininterrumpido de puntos dentro del rango de los valores numéricos de cualquier componente del ritmo o del sonido; respectivamente, frecuencia —pulso y altura—, amplitud —ataque e intensidad— y contenido armónico —color y micro-duración, además de las localizaciones que ocupe la materia musical —altura, lateralidad o profundidad<sup>3</sup>.

El discontinuo es un sub-conjunto del continuo y se entiende como una serie de puntos discretos extraídos del continuo, donde el valor numérico,  $n$ , de cualquiera de sus posibles componentes tiende, a diferencia del continuo, a ser finito.

### 2.2. Ámbito de referencia

La totalidad del rango comprendido por un componente se define como ámbito de referencia, mismo que depende de límites perceptivos, físicos, matemáticos u otros<sup>4</sup>.

### 2.3. Escala y términos

Se entiende por escala a la ordenación en secuencia de un conjunto finito de puntos discretos adyacentes establecidos dentro de un ámbito de referencia, donde cada punto es un término.

### 2.4. Escala inicial

Se entiende por escala inicial a un conjunto finito, fijo y ordenado de términos, lo que se identifica en el caso de los pulsos con un patrón rítmico, un compás o un valor rítmico.

### 2.5. Escala total

La escala inicial puede reproducirse de formas parcial, total o múltiple dentro del rango del ámbito de referencia. El conjunto de términos contenidos en dichas repeticiones se denomina escala total. La escala inicial es el conjunto generador y la escala total el conjunto completo.

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<sup>3</sup>Las ideas anteriores remiten a la noción de macro-timbre y se exponen de manera extensa en [4] y [6].

<sup>4</sup>El límite de las duraciones rítmicas más pequeñas se fija de acuerdo a la frontera perceptiva entre ritmo y sonido —entre 16 y 20 ciclos por segundo—, mientras que los máximos de duración carecen de una frontera precisa para diferenciar, por ejemplo, las duraciones rítmicas de las duraciones de una forma musical.

## 2.6. Dimensión de las escalas inicial y total

El número de términos de una escala inicial o total constituye su dimensión,  $D$ . Una escala inicial es idéntica en dimensión a la escala total de ser una escala única —escalas irreproducibles en otro registro, generadas por un grupo de instrumentos, etc.— o de ser difícil para su discriminación perceptiva —por ejemplo, debido a la inexactitud de la notación musical tradicional: “*ppp*”, “con sordina”, “*vibrato assai*”, etc.

La escala inicial puede alcanzar diversas dimensiones al expresarse en una escala total; por ejemplo, la escala total de la escala inicial diatónica es distinta en los registros de un piano, un violín o una obra musical. A su vez, una escala inicial de pulsos —un compás, por ejemplo— puede tener diversas dimensiones de escala total al expresarse, por ejemplo, en una forma musical, donde un tema, un movimiento o una obra completa pueden reproducir varias veces la dimensión de un mismo compás.

## 2.7. Registros de la escala total

La reproducción de una escala inicial sobre una escala total puede ocupar tantos registros como permita el ámbito de un componente dado o la percepción que pueda tenerse de éste.

# 3. Intervalos

## 3.1. Intervalo

Se entiende por intervalo la base de medida para medir la distancia entre dos términos cualesquiera de una escala. En matemáticas se denomina distancia a una función numérica que relaciona dos puntos de un espacio, concepto más cercano al del intervalo en música. Mientras, se entiende por intervalo matemático al conjunto total de puntos comprendidos entre dos extremos. El concepto de intervalo matemático equivale en música al intervalo continuo que abarca todos los puntos del ámbito por los que atraviesa. El continuo puede compararse, por ejemplo, con un intervalo de la recta real, lo que en una secuencia musical equivale al *glissando* y que en la vertical corresponde a la aglomeración sincrónica de todos los puntos de dicha trayectoria.

## 3.2. Orden de la escala

Al abstraer el valor numérico que en la realidad física tengan los términos o los intervalos de una escala, ésta puede expresarse como una sucesión ordenada de términos y ordenarse en números enteros que van de 0 a  $n$ :

$$0, 1, 2, \dots, n.$$

### 3.3. Unidad general de mínima distancia: $d1$

El valor numérico más pequeño para medir los intervalos es la unidad general de mínima distancia, que equivale a 1 y se expresa como  $d1$ . En lo sucesivo se utilizará dicha unidad para medir diversos tipos de distancia para construir un espacio discontinuo con la fluidez propia que caracteriza a las relaciones dentro del espacio continuo.

### 3.4. Mínima distancia entre intervalos

Se entiende por mínima distancia entre intervalos a toda aquella que denota la adyacencia entre los términos de una escala. En una escala inicial de dimensión  $D4$ , cualquier intervalo formado entre sus términos adyacentes se expresa como una mínima distancia  $d1$ :

$$D4 : \quad 0, \quad 1, \quad 2, \quad 3$$

### 3.5. Dimensión del intervalo

La dimensión del intervalo es la suma de mínimas distancias  $d1$  que separan a un término de cualquier otro dentro de una escala, inicial o total. Por ejemplo, en una escala inicial de alturas con dimensión  $D12$ , escala cromática de alturas, una selección arbitraria de distintos términos entre el 0 y el 11 dará los siguientes intervalos:

$$D12 : \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11$$

Los valores anteriores pueden igualmente observarse como pulsos al adoptar como mínima distancia  $d1$ , por ejemplo, una semicorchea. El conjunto de doce semicorcheas crea un compás de 12/16, dentro del cual se tienen los siguientes intervalos:

$$D12 : \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11$$

Los dos pares de intervalos creados en ambas escalas se expresan como una suma de mínimas distancias entre sus términos extremos:

$$\begin{aligned} [0, 2] &= d2 \\ [7, 11] &= d4 \end{aligned}$$

### 3.6. Espacio de operación

Para operar de modo abstracto con los intervalos se requiere reducir las dimensiones de cualquier escala total al espacio de la escala inicial. Por ejemplo, los intervalos de varias escalas totales con distintas dimensiones se reducen a los siguientes intervalos dentro de sus respectivas escalas iniciales:

$$D6 : 0, 1, 2, 3, 4, 5$$

$$121 - (6 \times 20) = 121 - 120 = 1$$

$$D10 : 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$$

$$43 - (4 \times 10) = 43 - 40 = 3$$

$$D12 : 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$$

$$17 - 12 = 5$$

$$D19 : 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18$$

$$159 - (19 \times 8) = 159 - 152 = 7$$

### 3.7. Intervalos de duplicación y del ciclo de la escala inicial

Dentro de la escala inicial, se entiende por intervalo de duplicación al conjunto de distancias comprendidas entre el primero y el último término. En el caso de los intervalos de duración y de altura dicha idea se entiende como intervalo de duplicación frecuencial (*idf*). El intervalo de duplicación es idéntico a la dimensión de la escala inicial (Figura 1):

$$D12 : 0, \dots, 12 = d12.$$

El ciclo de la escala inicial permite considerar que del último término se vuelve al primero mediante una mínima distancia  $d1$  entrabmos, misma que se denomina intervalo del ciclo de la escala inicial, relación inversa a la del intervalo de duplicación:

$$D12 : [11, 0] = d1.$$

## 4. Potencial combinatorio

### 4.1. Potencial combinatorio de las escalas

Toda escala posee un potencial combinatorio del conjunto de agrupaciones posibles de sus intervalos, de 1 en 1, de 2 en 2, ... hasta alcanzar la agrupación de todos los intervalos de dimensión 1.

### 4.2. Particiones no permutables

Las distintas agrupaciones de los intervalos de una escala generan una extensa combinatoria difícil de manipular en la práctica, lo que requiere reducirse a un conjunto mínimo de agrupaciones, las particiones no permutables del intervalo de duplicación.

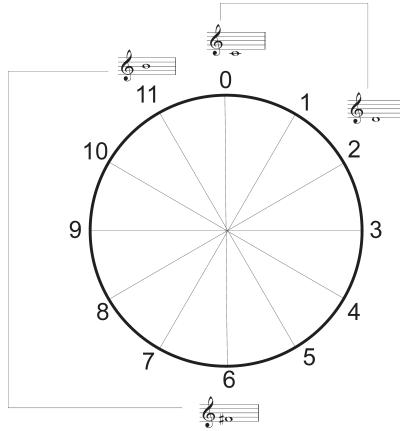


Figura 1: Ciclo de una escala inicial de altura, dimensión  $D_{12}$ . Se ilustran el intervalo  $[0, 2]$ , Do Re, y el intervalo del ciclo  $[11, 0]$ , Si Do.

### 4.3. Identidad de intervalos

El conjunto total de particiones no permutables de la dimensión  $D$  de una escala se denomina identidad de intervalos y se obtiene al ordenar de menor a mayor los intervalos producto de las particiones.

### 4.4. Potencial combinatorio de una escala

El conjunto total de identidades de intervalos de una escala constituye su potencial combinatorio, ordenable de acuerdo al número de niveles de densidad,  $N$ , que tenga una escala. Por ejemplo, una escala de dimensión  $D_6$  genera un conjunto total de 11 identidades en 6 distintos niveles (Figura 2).

El orden de los intervalos contenidos por las identidades es el punto de partida para generar su potencial combinatorio, a su vez producto del número total de permutaciones que generan los intervalos de una identidad. Matemáticamente, el potencial combinatorio coincide con los coeficientes multinomiales

$$\frac{E!}{r_1!r_2!\cdots r_x!}$$

donde  $r_i$  es el número de repeticiones de los términos de la identidad [14].

### 4.5. Órbita de la identidad

El conjunto ordenado de permutaciones a mínima distancia del potencial combinatorio de una identidad constituye la órbita de la identidad.

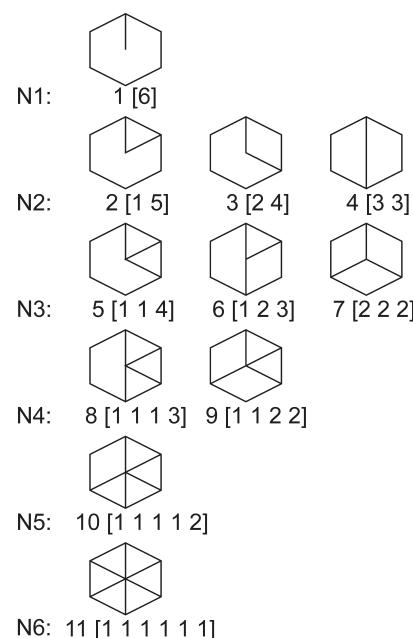


Figura 2: Particiones de una escala de dimensión  $D6$ . El conjunto total de identidades se reduce a 11 elementos, no permutables, mismos que aparecen ordenados de acuerdo al nivel de densidad  $N$  en número de intervalos, de la identidad 1 [6],  $N1$ , a la identidad 11 [1, 1, 1, 1, 1, 1],  $N6$ .

#### 4.5.1. Estructura cíclica de las identidades

La permutación a mínima distancia  $d1$  de los intervalos contenidos por una identidad conduce al orden inicial de la misma. Por ejemplo, una identidad de tres términos distintos entre sí  $[a, b, c]$  genera un ciclo de 6 permutaciones, donde cada una se encuentra a  $d1$  de las que le son más próximas:

$$\rightarrow [a, b, c] \leftrightarrow [b, a, c] \leftrightarrow [b, c, a] \leftrightarrow [c, b, a] \leftrightarrow [c, a, b] \leftrightarrow [a, c, b] \leftarrow$$

#### 4.5.2. Red

La noción de red proviene de la teoría matemática de grafos, empleada aquí para representar el orden de los intervalos de las escalas. Las redes de las identidades están constituidas por un número limitado de combinaciones gráficas dentro de una distribución simétrica; a saber, líneas, hexágonos o hexágonos ligados a cuadrados.

### 4.6. Permutaedro

Las permutaciones de la órbita de la identidad generan una red cuya estructura combinatoria ofrece un orden geométrico similar al de los cristales. Una nueva área de la matemática combinatoria denomina a dicha estructura retal como combinaedros [16], que en la teoría de los intervalos se reduce a la denominación de permutaedro, nombre asociado a la estructura geométrica de los ciclos que genera la permutación de los intervalos.

#### 4.6.1. Permutaedro inicial

El ciclo inicial de permutaciones a mínima distancia  $d1$  de una identidad se denomina permutaedro inicial. El caso de la identidad de tres términos distintos entre sí arriba señalado se ilustra a continuación mediante un hexágono en el que se representan los ciclos de permutaciones de tres distintas identidades de altura dentro de la escala  $D12$ ; el contenido de intervalos de las identidades que sirven de ejemplo varía gradualmente de lo consonante a lo disonante (Figura 3).

#### 4.6.2. Permutaedro total

El conjunto de transposiciones de la identidad a los términos de la escala genera un permutaedro en el que la repetición sucesiva del ciclo inicial de las permutaciones se reproduce en cada una de las alturas de dicha escala para formar una red conectiva constituida por relaciones a mínima distancia  $d1$ , en donde cada punto es un nodo. La identidad de alturas  $[1, 5, 6]$  ilustra el permutaedro total, en este caso formado por un conjunto de 36 hexágonos que tienen como núcleo una altura específica (Figura 4).

Toda identidad de intervalos o cualquiera de sus permutaciones se entiende como una acumulación de términos contiguos sobre un continuo lineal. Al efectuar la permutación sólo entre los intervalos adyacentes de una identidad, ésta

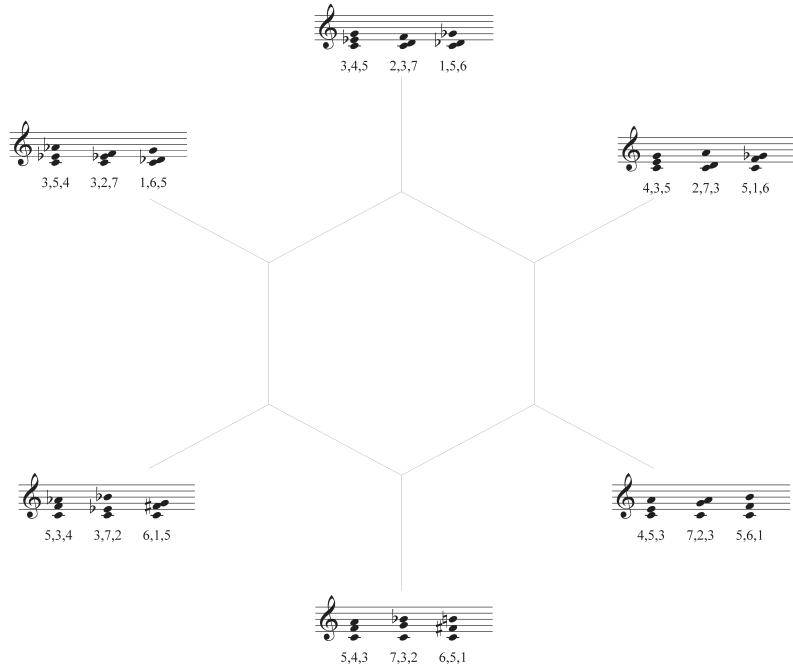


Figura 3: Hexágono del permutaedro de tres identidades de la escala de altura  $D_{12}$ , donde cada identidad —arriba al centro— contiene 3 intervalos distintos entre sí:  $[3, 4, 5]$ ,  $[2, 3, 7]$ ,  $[1, 5, 6]$ . Al observar el caso de la primera de las tres identidades,  $[3, 4, 5]$ , se tiene que sus permutaciones, en una lectura de las aristas del hexágono, responden a todas las posibilidades que ofrece la armonía consonante tradicional: acordes menor y mayor, ambos en estado fundamental, primera inversión en modo menor, segunda inversión en mayor y en menor y primera inversión en modo mayor. Dicha relación permite considerar bajo una nueva perspectiva los casos de las otras dos identidades, de mediana disonancia  $[2, 3, 7]$ , y de mayor disonancia  $[1, 5, 6]$ , para lo cual es útil observar la regularidad de los ámbitos dentro de los que quedan comprendidos los registros de sus 6 permutaciones:  $[2, 3, 7]$ , tres pares de permutaciones ordenados por cuarta justa [5], sexta mayor [9] y séptima menor [10], todos ellos intervalos que complementan a los intervalos contenidos por la identidad, o sea  $[5, 7]$ ,  $[3, 9]$  y  $[2, 10]$ ; lo mismo ocurre con la identidad  $[1, 5, 6]$ : quinta disminuida [6, 6], quinta justa [5, 7] y séptima mayor [1, 11].

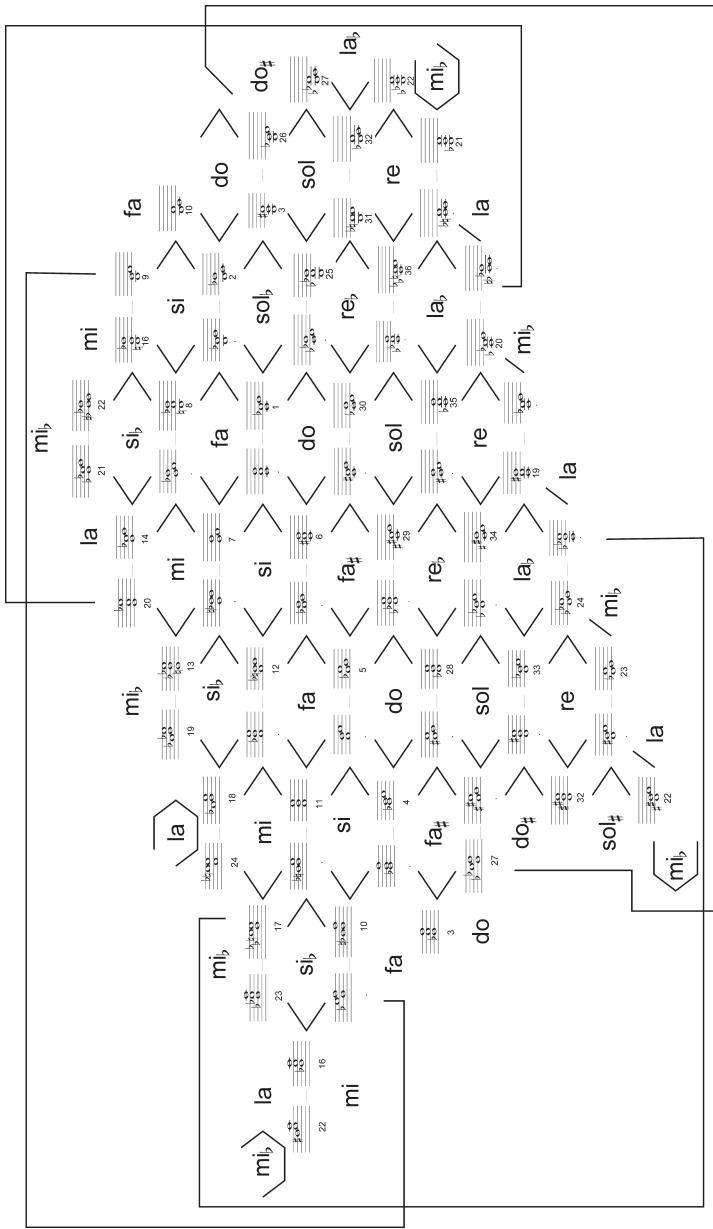


Figura 4: Permutaedro total de la identidad  $[1, 5, 6]$ , escala  $D12$  de alturas. La identidad aparece en la quinta línea hacia el centro derecha y a la derecha del hexágono 1, Do, Re $\flat$ , Sol $\flat$ , y la última transposición dos líneas más abajo y a la derecha de la anterior, en el hexágono 36, La $\flat$ , Re $\flat$ , Re.

puede vincularse con todas las permutaciones posibles mediante el intercambio continuo de sus intervalos:

$$[a, b, c, d] \rightarrow [a, c, b, d] \rightarrow [c, a, b, d] \rightarrow [c, a, d, b] \rightarrow \text{etc.}$$

La creación de un vasto espacio conectivo entre los intervalos de una escala, entre las identidades de su potencial combinatorio y entre identidades de potenciales combinatorios de distinta dimensión constituye el continuo de las escalas<sup>5</sup>. La posibilidad de reducir cualquier agrupación de intervalos a una identidad así como la riqueza de la estructura retal del permutaedro ofrecen una óptima ordenación del universo discontinuo en música para su manejo libre y eficaz en creación, análisis, pedagogía, musicología y etnomusicología.

## 5. Teoría *d1*, MúSIIC-Win

La teoría *d1* se origina entre 1980 y 1990 al ser la base de mis cursos de teoría de la creación musical en la Escuela Nacional de Música, UNAM, y de los seminarios de posgrado impartidos en la Universidad de California en San Diego. La teoría es a su vez la primera de dos partes de una tesis doctoral concluida en 1995, misma que se apoya en los primeros resultados del programa informático Teoría *d1*, MúSIIC-Win<sup>6</sup>, cuyo inicio data de 1990 [7].

La filosofía de la teoría *d1* reside en aportar sustento matemático al estudio objetivo de los elementos de la música, el ritmo y el sonido, por fuera de toda inclinación estética, lo que reivindica a la noción de teoría como un territorio independiente de la noción de sistema musical de composición, donde

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<sup>5</sup>El potencial combinatorio de una escala, la identidad o el permutaedro son la referencia para acceder a materiales de otras escalas mediante mínimas distancias *d1*; es decir, cuando el número de términos sea a su vez adyacente a la estructura que sirva de punto de partida —potencial, identidad o permutaedro.

<sup>6</sup>El programa [4] es un producto del proyecto PAPIME, DGAPA, ENM, UNAM, 1997-2000, asignado al Laboratorio de Creación Musical (LACREMUS) a cargo de Julio Estrada en la Escuela Nacional de Música. La primera fase de esta búsqueda se inició en 1990 con un proyecto PAPIIT, DGAPA, IIEs, UNAM, para desarrollar MúSIIC-MSDOS en el Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas (1990-97), innovación tecnológica producto de la interacción de grupos de investigadores en música, matemáticas y computación. El diseño, ingeniería de software y programación de Teoría *d1*, MúSIIC-Win, bajo el sistema Windows® —una marca registrada de Microsoft Corp.—, fueron realizados por Max Díaz, Erik Schwarz y Víctor Adán, bajo la dirección de Julio Estrada. Los requerimientos mínimos para operar con el programa son computadora personal con Windows 95, o más reciente; memoria RAM: 24Mb; espacio libre en disco duro: 10Mb; computadora multimedia: tarjeta de sonido y bocinas, o bien, tarjeta MIDI con sintetizador y un amplificador externo opcional. Con una apariencia totalmente rediseñada, y estándar al ambiente Windows®, MúSIIC-Win permite una exploración e interacción intuitivas para el usuario, una visualización en notas musicales así como la conversión de éstas a pulsos o a alturas mediante una computadora “multimedia” o una interface MIDI. Al tratarse de un programa en desarrollo, todas las alteraciones de las notas aparecen indicadas con el signo de #, un aspecto que el desarrollo a futuro del programa contempla amplificar al incluir signos de bemol. Otros desarrollos permitirán la proyección de cualquier caso de escala integrada por intervalos desiguales, a la vez que una amplificación de los potenciales para acceder a escalas de hasta 48 términos, límite donde la sensación auditiva de intervalo de altura se diluye y cede su sitio a la sensación perceptiva de un continuo.

los objetivos tienden a organizar los materiales musicales para su empleo en la práctica creativa. Dicha filosofía se refleja en el programa informático Teoría *d1*, MúSIIC-Win, herramienta tecnológica de búsqueda creativa, pedagógica o musicológica que facilita y extiende las investigaciones sobre la teoría del potencial combinatorio de los intervalos en música, al ofrecer la proyección de escalas de altura y de duración constituidas por 3 a 24 términos expresadas como divisiones iguales del intervalo de duplicación de frecuencia (*idf*). El programa permite generar igualmente formas de onda rítmicas y sonoras al proyectarlas como identidades de intervalos, cuya permutación independiente de pulsos y de amplitudes dentro del límite de 3 a 24 términos ofrece una fórmula teórica y tecnológica novedosa para abordar a nivel micro la forma de onda [9].

Un permutaedro del tipo  $[a, b, b, c, c, c]$  en el nivel medio, *N6*, de la escala *D12*, ilustra las facilidades del programa MúSIIC-Win (Figura 5).

## 6. Aplicaciones al análisis de la teoría *d1*

### 6.1. Clases de intervalo

La teoría *d1* procura un aparato preciso para conocer y reconocer las distintas agrupaciones de intervalos contenidas en secuencia o en forma vertical en música creada a partir de escalas de duración o de altura. La base de la teoría *d1* son las clases de intervalo, en oposición a las clases de altura que predominan en la composición musical de los siglos XX y XXI [10]. La teoría *d1* es a su vez una nueva herramienta para la etnomusicología, donde numerosos sistemas de tradiciones antiguas basadas en escalas provenientes de usos vocales o instrumentales pueden contar con la referencia objetiva del potencial combinatorio de los intervalos<sup>7</sup>. Un aspecto novedoso de la teoría es la integración del ritmo y del sonido dentro de un mismo espacio, al entender que ambos son parte de un mismo continuo de frecuencias, un tema que abordan con gran anticipación los trabajos de Henry Dixon Cowell [2]. La amplitud del espectro de la teoría *d1* es igualmente aplicable al análisis de música escrita de tradición europea en los materiales basados en escalas o en series de duración, como las secuencias de duraciones o los compases [4, 17], incluso en el caso de micro-divisiones, como desarrolla la obra de Conlon Nancarrow a partir de las proposiciones teóricas de Cowell [2] y con los *multi-tempi* [5] o en escalas de altura, como las de orden modal, tonal, cromático, “sin octava” o *idf*, de macro-intervalos —escalas cuya división del *idf* es menor de 12 términos— [15] o de micro-intervalos —divisiones mayores de 12 términos, como Carrillo [1] o Haba [11].

<sup>7</sup>Entre otros, considérense los ejemplos de la música de la India mediante el estudio del potencial combinatorio de los talas rítmicos o, como en el trabajo pionero de Jairazbhoy, creador del primer permutaedro en música como resultado de su estudio de las escalas de altura del Norte de la India, los thats: [12].

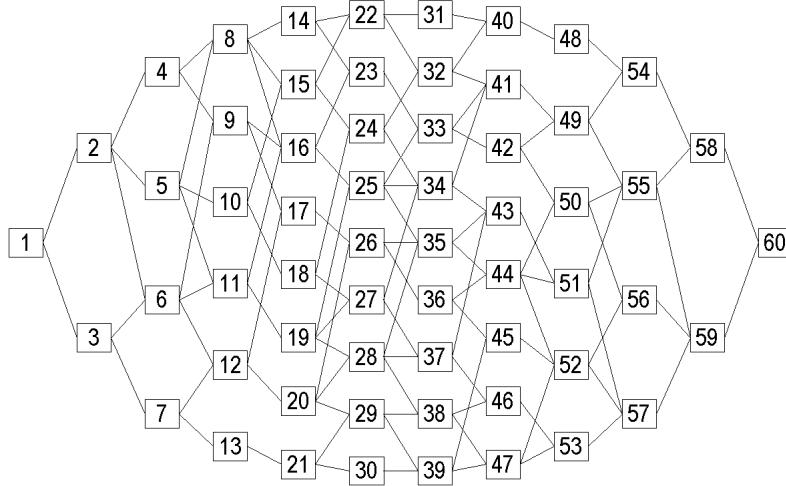


Figura 5: Permutaedro de una identidad de intervalos del tipo  $[a, b, b, c, c, c]$ , caso de las identidades 52 [1, 1, 1, 2, 2, 5] y 55 [1, 1, 2, 2, 2, 4], de nivel  $n6$ , escala  $D12$ , grafo generado por el programa teoría *d1*, MúSIC-Win. Las 60 permutaciones aparecen numeradas; la identidad aparece a la izquierda con el número 1. Los vínculos a mínima distancia  $d1$  entre las operaciones son las líneas que unen una permutación con otra y crean la red. Ésta muestra una secuencia ordenada entre cuadrados y hexágonos, cuyo recorrido de izquierda a derecha por la parte inferior del grafo es el siguiente: 2 cuadrados 1, 2, 6, 3; 3, 6, 12, 7; 1 hexágono 7, 12, 20, 29, 21, 13; 3 cuadrados 21, 29, 39, 30; 29, 38, 47, 39; 38, 46, 53, 47; 1 hexágono 53, 46, 37, 43, 51, 57; 3 cuadrados 52, 56, 59, 57; 59, 56, 50, 55; 59, 55, 58, 60. La simetría de la misma secuencia se aprecia al recorrer el grafo por encima y partir de la permutación 60 a la identidad. La red permite explorar los posibles caminos entre una operación y otra; por ejemplo, la vía más corta entre 1 y la permutación 60 —centrada a la derecha— requiere un mínimo de 11 pasos, mismos que pueden recorrerse al adoptar cualquiera de las posibles direcciones que las unen. Las relaciones entre distintas sonoridades es un tema común en música, un tema en el que las redes pueden contribuir a estudiar de manera más sistemática.

## 6.2. Potencial combinatorio de la escala de alturas $D12$

Las diversas aplicaciones de la teoría  $d1$  no exigen mayor conocimiento matemático que la aptitud para reducir los intervalos a números enteros, uso corriente en la escala de 12 alturas (por ejemplo, 1, semitono, 2, tono, … 12, octava), siendo el caso más general en la música escrita de tradición europea y que la teoría  $d1$  representa mediante un potencial combinatorio que abarca en total 77 identidades de intervalos, desde un solo intervalo [12] hasta la agrupación de los 12 semitonos [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1], equivalente a la escala de dimensión  $D12$  (Figura 6).

La comprensión del potencial combinatorio de la escala de alturas  $D12$  se facilita al dividirlo en cuatro zonas principales de acuerdo con su contenido en intervalos y su densidad:

- A. inferior izquierda: intervalos de mayor dimensión y menor densidad
- B. superior izquierda: intervalos de menor dimensión y menor densidad
- C. inferior derecha: intervalos de mayor dimensión y mayor densidad
- D. derecha: intervalos de menor dimensión y mayor densidad.

Cada zona es identifiable por su referencia general a grandes sistemas, como espacios musicales consonante (A.), disonante (B.), atonal extendido (C.) y de masas (D.). Desde esa perspectiva el potencial combinatorio sugiere la exploración colectiva del oído musical a través del avance histórico en un proceso que va de la simple monodia —identidad 1 [12]— al no menos simple cluster —identidad 77 [1,1,1,1,1,1,1,1,1,1,1,1]—, ambos casos con una combinatoria nula.

Más allá de suponer que sólo se comparte un lenguaje, la evolución constante de la escritura hacia los máximos de densidad invita a comprender la importancia musical del fenómeno perceptivo a través de la transmisión inadvertida de la experiencia auditiva, en la cual la memoria y el cálculo contribuyen de manera definitiva a la adquisición de sonoridades nuevas desde sonoridades conocidas. Éstas se vinculan con sus vecinas de mayor densidad mediante transiciones inadvertidas a mínima distancia,  $d1$ , una idea que se evidencia al abordar la música escrita de tradición europea en los albores del siglo XX, cuyos rupturas históricas son sólo aparentes de considerar su evolución como una serie continua de mínimos cambios dentro del potencial combinatorio, como ilustra la división en 17 zonas (Cf. Figura 6).

La síntesis que ofrecen los intervalos contenidos por las identidades conduce a observar la noción de sonoridad resultante como una información auditiva que transita de una época a otras bajo denominaciones incluso distantes. En el fondo, el reconocimiento intuitivo de una agrupación de intervalos forma parte de la experiencia con que cuenta el oído creador al explorar nuevas sonoridades resultantes, un asunto que puede apreciarse en el análisis musical que ocupa las siguientes páginas.

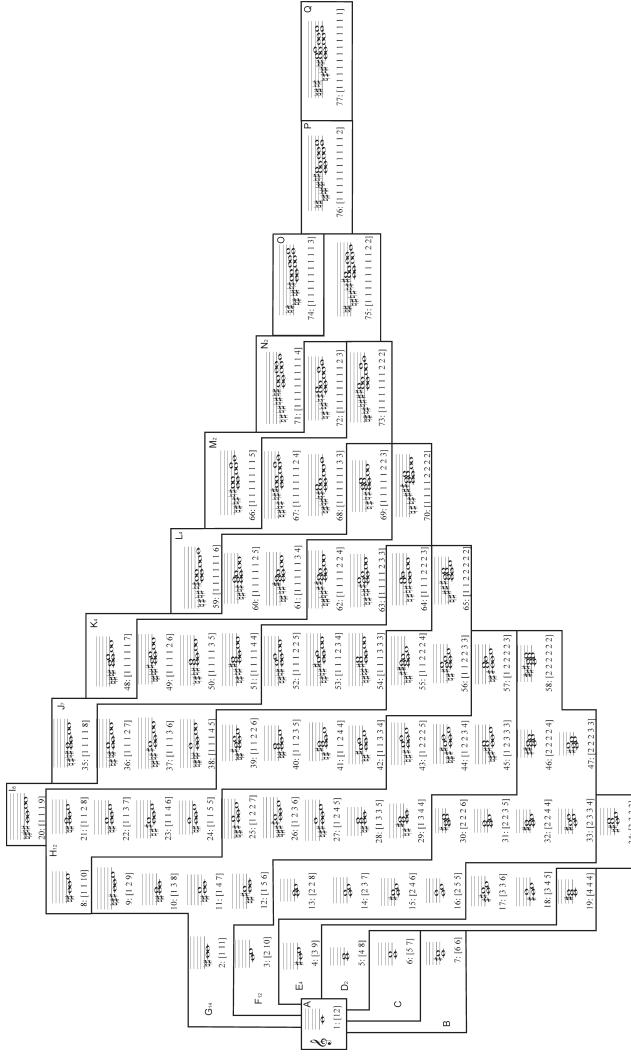


Figura 6: Potencial combinatorio de la escala D12; las identidades se enumeran de la 1 a la 77. A la derecha, la identidad 1 [12]; el incremento en intervalos de las identidades denota un ensanchamiento gradual del grafo hasta la cuarta columna; a partir de la quinta, la densidad desciende hasta la duodécima, identidad 77. La imagen agrupa las identidades en 17 zonas, A a Q, de acuerdo con su contenido en intervalos. Las zonas A, B y C corresponden a las identidades 1 [12], 6 [5, 7] y 7 [6, 6], y quedan aisladas al no derivar de ellas ninguna identidad; mientras, de la zona D, identidad 5 [4, 8], deriva la identidad 19 [4, 4, 4]; de la zona E, identidad 4 [3, 9], derivan 3 identidades, 17 [3, 3, 6], 18 [3, 4, 5] y 34 [3, 3, 3]. En contraste, las zonas H e I, de singular importancia en el análisis de las *Seis piezas* de Schoenberg, tienen como puntos de partida las identidades 8 [1, 1, 10] y 20 [1, 1, 1, 9], que conducen, respectivamente, a la identidad 65 [1, 1, 2, 2, 2, 2, 2], cuyo contenido en intervalos coincide con la escala diatónica, y a la identidad 64 [1, 1, 1, 2, 2, 2, 3], cuyo contenido en intervalos coincide con la escala menor armónica.

### 6.3. Método de análisis musical

Al prescindir de los criterios propios de la organización de cada sistema musical, los análisis a los que conduce la teoría *d1* se basan en el estudio de datos neutros sobre los cuales construir con objetividad una interpretación nueva. En ésta, los preceptos compositivos ceden su sitio a la noción de identidad, aplicable a un amplio conjunto de estructuras tales como la de agrupación en secuencia o la de agrupación vertical, casos en los que la noción misma de permutaedro incita a revisar la vieja noción de inversión armónica, lo mismo que la caducidad de la armonía consonante (Cf. Figura 3).

Al detectar las identidades presentes en la evolución de una obra así como la riqueza que puedan contener sus respectivos permutaedros, el método de análisis observa la conectividad retal entre las identidades mediante grafos que proyectan las relaciones secuenciales que genera el sistema particular de cada obra. Dicha idea contrasta con la tendencia generalizada a analizar distintos casos de música como si fuesen el producto del sistema en el que aparentemente se basan, lo cual sólo permite afirmar o negar la pertenencia a su origen. En contraste, la proyección de mapeos de las identidades detectadas en el análisis de una obra aporta un método original para representar la búsqueda individual o colectiva en el ámbito del potencial combinatorio de la escala. El análisis de dichas proyecciones conduce a su vez a observar el espacio que ocupa cada experiencia musical con objeto de compararlo con el espacio total, de donde la noción de exploración del potencial combinatorio de las escalas.

La metodología de análisis de la teoría *d1* observa los intervalos contenidos en la música de cualquier época, apertura que permite incluir sin trabas al rico universo disonante, ajeno a la noción de consonancia y por ello tradicionalmente excluido de la discusión académica de las teorías armónicas. Éstas son con mucho más amplias que el universo consonante, una pequeña fracción del potencial combinatorio de la escala de alturas *D12*. Los análisis basados en la teoría *d1* prescinden de toda denominación que intente justificar las formas de asociación entre alturas, como ocurre con las “apoyaturas”, “notas de paso”, “anticipaciones” u otras propias de la tonalidad, lo mismo que todas aquellas nociones que, de manera particular, puedan derivar de los lenguajes modernos como la tonalidad extendida, la poli-tonalidad, la atonalidad, el serialismo, etc., lo cual remite a la exigencia permanente de objetividad en sus evaluaciones.

El método a seguir consiste inicialmente en reducir al ámbito de la escala inicial todas las agrupaciones de intervalos presentes en cada distinto momento de una partitura, para así obtener todas las identidades o las permutaciones de las mismas en la obra. El procedimiento se ilustra enseguida con un pasaje del madrigal a cinco voces *Moro, lasso, al mio duolo*, de Carlo Gesualdo (1566-1613), con el cual se facilita el abordaje de la música producida con la escala diatónica, expresada dentro de la escala cromática; en particular, cuando el autor es un precursor del cromatismo musical. La reducción del pasaje al ámbito de la escala inicial indica las mínimas distancias *d1* que habría que recorrer para transitar entre acordes contiguos en la partitura original (Figura 7). La evolución del pasaje anterior muestra la tendencia en la obra a conectar sonoridades

vecinas en número de intervalos —un intervalo menos, un intervalo de más o el mismo número de intervalos. La siguiente red conectiva representa el mismo pasaje, eliminando las permutaciones para resaltar tan sólo las identidades de los intervalos<sup>8</sup> (Figura 8).

## 7. Schoenberg, *Seis piezas para piano*, opus 19

La aproximación al análisis mediante la teoría *d1* se concentra en la detección de la verticalidad, uno de los temas que ocupan la atención de la musicología moderna al estudiar las relaciones armónicas. Las *Seis piezas para piano* (1913) de Arnold Schoenberg (1874-1951) revelan una exploración auditiva inédita del autor al tender de manera gradual a internarse del sistema tonal a un sistema aparentemente opuesto a éste, la atonalidad, e incluso a iniciar una inmersión inesperada en agrupaciones de alta densidad. Cada pieza constituye una referencia para observar dicha búsqueda hacia los límites mediante un sondeo novedoso de la combinatoria de la escala. Las *Seis piezas* se caracterizan por la brevedad y la economía de medios, inquietudes del momento que Schoenberg comparte con Anton Webern, algo que incita a entender que una parte de la empresa creativa reside en explorar las posibilidades armónicas de la escala, y que en la indagación schoenbergiana se apoya en la creación de una variedad de texturas en cada pieza, intento intuitivo y propositivo para abordar la relación espacio-temporal.

Para entender la búsqueda realizada en la obra desde la perspectiva de la exploración de la escala, el procedimiento a seguir consiste en describir someramente la textura de cada pieza, en reducir a identidades de intervalos cada aglomeración vertical detectada como una rebanada de la evolución secuencial y en representar mediante una red las identidades detectadas, sus niveles de densidad y tendencias conectivas.

### 7.1. Pieza I, polifonía libre

Los giros caprichosos de la melodía principal, acentuados por el carácter ternario del compás en 6/8 (Figura 9), se contrastan con pequeños elementos melódicos que, en su conjunto, evocan de manera fragmentaria la escala menor armónica mediante breves arpegios y algunos acordes que participan igualmente en una formación textural que intenta enfatizar las contradicciones entre dos tonalidades, como se aprecia en el tercer compás del fragmento arriba expuesto.

La red refleja una exploración extensa —cuantitativamente la mayor en toda la obra—, casi declarativa de las sonoridades resultantes y de la combinatoria conocida por el autor; al menos al inicio del proceso. Las 34 identidades que contiene, casi la mitad del potencial de la escala D12, cubren casi por entero los 6 niveles de densidad que abarcan. La tendencia propia de la textura polifónica

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<sup>8</sup>La representación en redes de materiales musicales se expone en varios trabajos anteriores; entre otros [8].

D. (1) (1) (2) (1) (3) (2) (4) ... (1) (1) (1) (1) (2) (1) (1) (1)

C.I. (12) (3.9) (4.8) (3.4,5) (3.4,1.4) (5.4,3) (3.5,4) (5.4,3) (5.3,4) (4.3,5) (2.5,5) (3.5,4) (5.4,3) (5.3,4) (4.3,5)

N. 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14

V. { 13

D. (1) (2) (3) (5) (2) (2) (2) (3) (1) (1)

C.I. (2.5,5) (3.5,4) (4.3,5) (4.5,3) (4.3,5) (3.5,4) (2,1,7,2) (3.5,4) (3.2,4,3) (3.5,4) (4.5,3)(4.3,5) (5.3,4)

N. 15 16 17 18 19 20 21 22 23 24 25 26 27

V. { 25 25

Figura 7: Gesualdo, fragmento de *Moro, lasso, al mio duolo*. En el sistema inferior aparece la partitura original, cuya evolución continua de agrupaciones verticales se enumera desde el inicio  $(0, 1, \dots, n)$ . En el sistema superior se anota la reducción de cada agrupación vertical a su respectiva identidad o permutación, en ambos casos cifradas entre paréntesis al pie de la pauta. Las cifras entre paréntesis por encima de las notas refieren a la cantidad de mínimas distancias *d1* por recorrer dentro de un permutaedro para alcanzar la siguiente sonoridad, lo que se representa como una permutación de la identidad, como la identidad o como una nueva identidad o alguna de sus permutaciones. Las notas negras cuadradas con ligaduras punteadas intentan representar los movimientos figurados que harían las voces para pasar de manera continua de un acorde al siguiente dentro del pasaje original. Por ejemplo, al pasar del primer al segundo compás, del acorde de Re mayor al acorde de Fa♯ mayor, podría suponerse como punto intermedio el acorde de Fa♯ menor, a mínima distancia *d1* de ambos acordes.

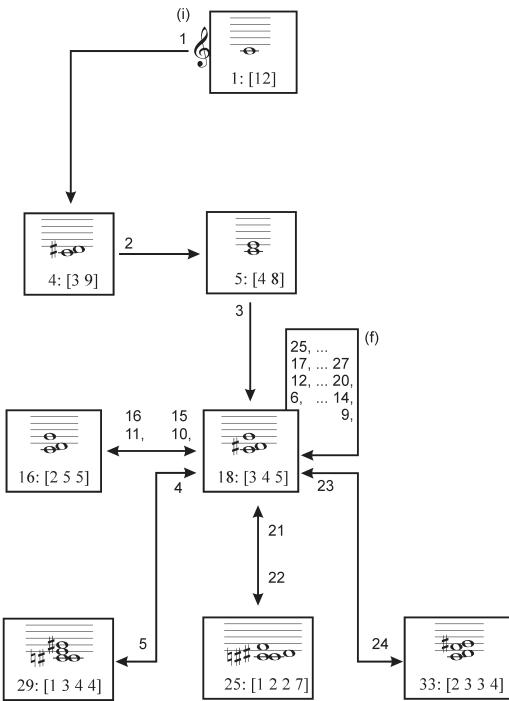


Figura 8: *Moro, lasso, al mio duolo*, Gesualdo, red conectiva entre 8 identidades de intervalos con niveles de densidad entre identidades de 1 a 4 intervalos; 27 transiciones en total. Las cifras vinculadas a las flechas de la red señalan el número de la transición. Las identidades 29, 25 y 33 al pie del grafo tienen igual combinatoria al poseer 3 intervalos distintos entre sí y uno repetido:  $[a, b, c, c]$ , casos que generan todos un permutaedro integrado por doce nodos (con objeto de facilitar al lector la consulta de cualquiera de dichas identidades en el programa Teoría d1, MúSIIC-Win arriba citado, todas las alteraciones de las notas aparecen con signos de  $\sharp$ ). La red sintetiza el micro-sistema del pasaje y muestra los límites de una búsqueda en el espacio tonal. Se distinguen 4 identidades consonantes  $[12]$ ,  $[3, 9]$ ,  $[4, 8]$ ,  $[3, 4, 5]$  —eje de la conectividad—, dos identidades propias de la armonía disonante  $[1, 3, 4, 4]$  —acordes de séptima mayor—,  $[2, 3, 3, 4]$  —acordes de séptima menor—, y dos más que el sistema tonal ensaya justificar con denominaciones como las mencionadas algo más arriba:  $[2, 5, 5]$  y  $[1, 2, 2, 7]$ . Esta última identidad muestra una aglomeración de cuatro alturas contiguas en la escala inicial, lo que permite vincular su sonoridad resultante a la música de épocas más recientes.

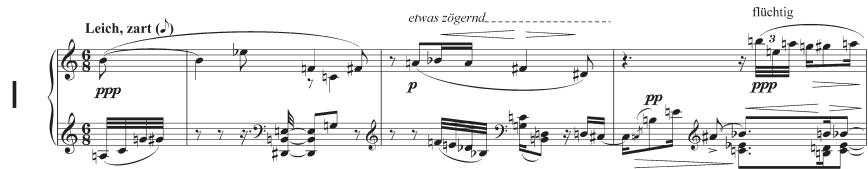
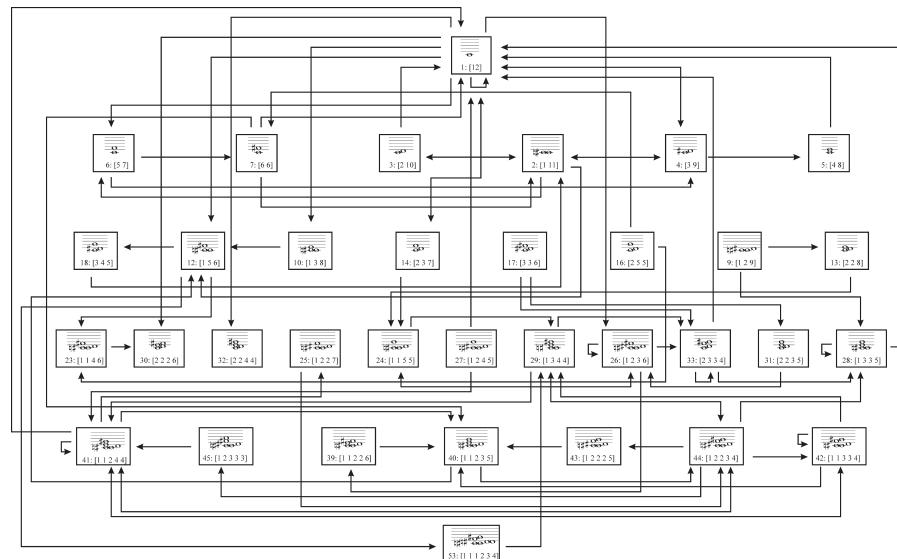
Figura 9: Schoenberg, *Seis piezas para piano*, fragmento inicial de la *Pieza I*.Figura 10: Schoenberg, *Pieza I*, red entre 34 identidades en un continuo de niveles que abarca del  $N1$  al  $N6$ .



Figura 11: Schoenberg, *Seis piezas para piano*, fragmento inicial de la *Pieza II*.

libre se manifiesta en el tejido de conexiones, que mantiene siempre la continuidad dentro de un mismo nivel —obsérvese en las conexiones horizontales— a la vez que relaciona identidades de distintos niveles, incluso extremos —por ejemplo, los saltos entre la identidad 12 [1, 5, 6] y la 53 [1, 1, 1, 2, 3, 4], o entre la 41 [1, 1, 2, 4, 4] y la 1 [12]. El grafo destaca la conectividad de la identidad 1 [12] con casi la mitad de las identidades contenidas en la red mediante un patrón que se distingue por el salto. La discontinuidad en la vinculación de algunas sonoridades es un indicio de la evolución posterior de la obra, de observar que reaparece en las *Piezas III y IV*, un aspecto que da énfasis al hallazgo sorpresivo propio de la búsqueda, aunque la exploración no exceda aún el límite de sonoridades resultantes que derivan de la identidad con mayor número de semitonos, identidad 53 [1, 1, 1, 2, 3, 4] de nivel *N6* —parte inferior de la red—, que pertenece al espacio de la zona I del potencial, la escala menor armónica, identidad 64 [1, 1, 1, 2, 2, 2, 3] de nivel *N7*, de la cual derivan 56 identidades (Cf. Figura 6), y cuya variedad de intervalos incrementa la combinatoria, un dato que puede constatarse al observar la exploración de la obra.

## 7.2. *Pieza II*, intervalo-objeto

Un breve motivo rítmico en *staccato* deja escuchar con persistencia una tercera mayor, sonoridad resultante que gravita en la pieza y deviene un intervalo-objeto, que en el contexto de la armonía disonante remite al oído a la brillantez del acorde mayor. La textura juega con las relaciones posibles de dicho intervalo dentro de las tonalidades que le son vecinas, como se escucha desde el inicio, que parece estar en Sol mayor, Mi menor y Do menor, con lo cual genera una frecuente ambigüedad dentro del universo tonal al tiempo que evoca la fórmula de la pieza anterior de emprender la búsqueda a partir de una sola altura —fórmula que aquí se percibe como una variación de color.

La tercera mayor inicia la pieza (i), ocupa el centro de la conectividad y aporta desde el principio una sonoridad hasta ahora no escuchada —dos terceras mayores superpuestas, el acorde aumentado, identidad 19 [4, 4, 4]— además de una sonoridad ya advertida, la superposición de dos segundas mayores que se suman en otra tercera mayor, identidad 13 [2, 2, 8]. La exploración divide en dos los espacios de los niveles *N1* a *N3* y *N5* a *N6*, ensayo que evoca sonoridades resultantes conocidas de la tradición en contraste con otras donde los intervalos

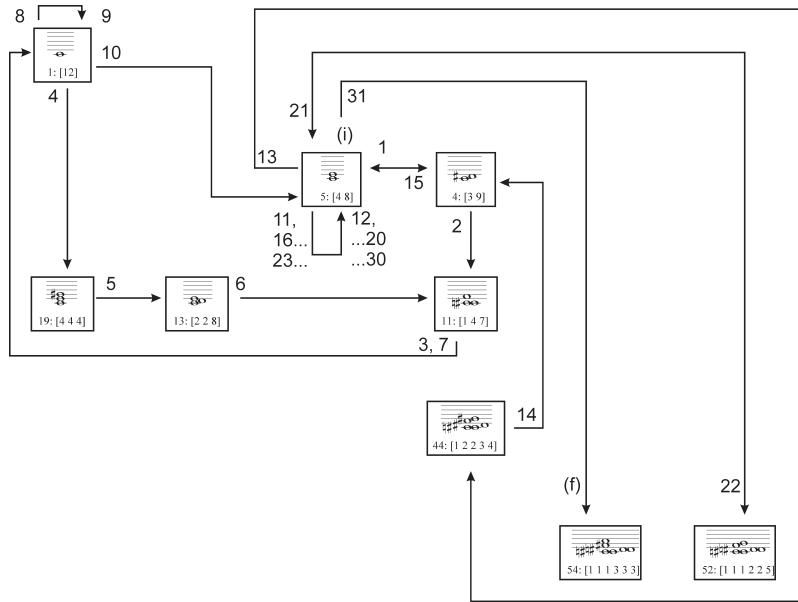


Figura 12: Schoenberg, *Pieza II*, red entre 9 identidades en un espacio discontinuo de niveles que abarca del *N1* al *N3* y del *N5* al *N6*.

se aglomeran verticalmente. Dos sonoridades de nivel *N6* hasta entonces no escuchadas en la obra recalan la intención de llegar poco a poco al límite de mayor densidad, con lo cual se alcanza también el mayor número de semitonos, las identidades 52 [1, 1, 1, 2, 2, 5] y 54 [1, 1, 1, 3, 3, 3], final (f) de la pieza, que aún no abandona la tonalidad al derivar ambas de la identidad 64 [1, 1, 1, 2, 2, 2, 3], zona I del potencial.

### 7.3. *Pieza III, coral*

La textura en el estilo de un coral resalta al doblar el bajo, en *pp*, cuya melodía al inicio desvela un color tonal que avanza de Mi♭ menor a Fa menor; mientras, la parte superior ofrece una evolución hecha de acordes densos en *ff*,

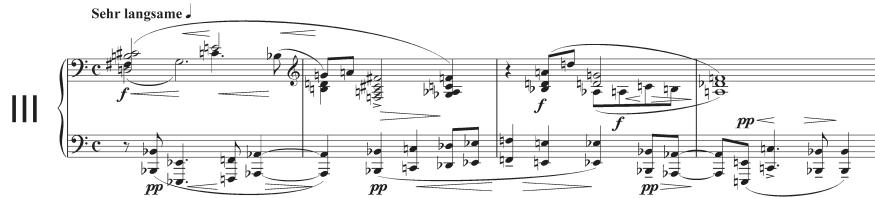


Figura 13: Schoenberg, *Seis piezas para piano*, fragmento inicial de la *Pieza III*.

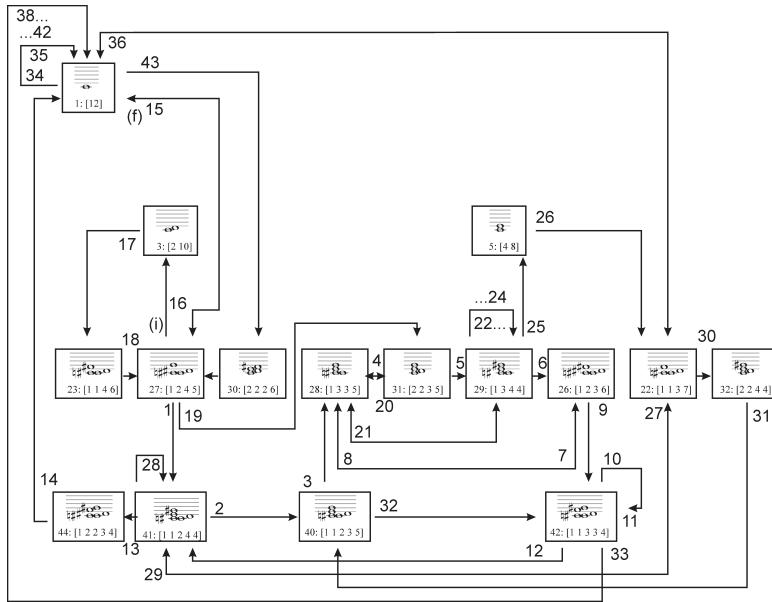
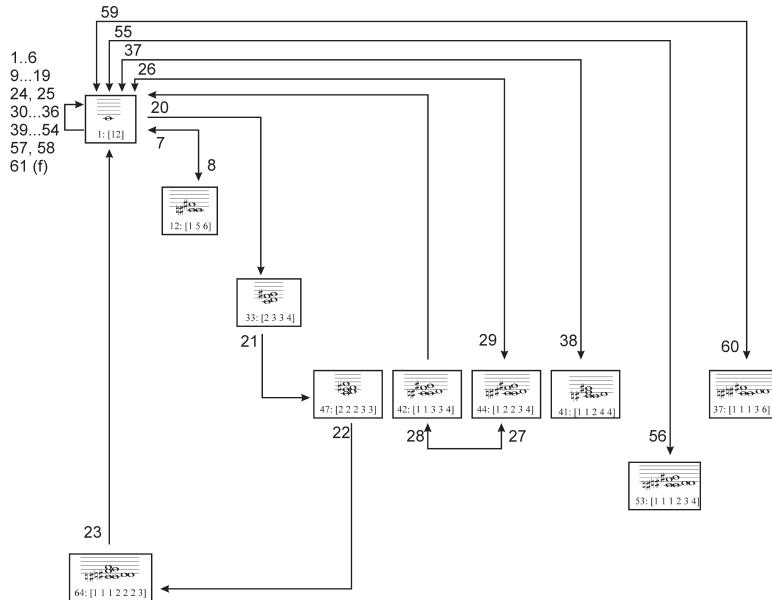


Figura 15: Schoenberg, *Seis piezas para piano*, fragmento inicial de la Pieza IV.Figura 16: Schoenberg, Pieza IV, red entre 10 identidades en un espacio discontinuo que comprende los niveles  $N_1$  y del  $N_3$  al  $N_7$ .

alusión continua a tonalidades distintas, como la evolución melódica inicial, que pasa de Si♭ menor a Do mayor, La mayor, etc. Los frecuentes encuentros de la melodía con acordes que surgen de manera inesperada dan en este caso la idea de tropiezo y, por la contradicción entre materiales consonantes proveniente de una poli-tonalidad que recuerda al raro diálogo entre el bajo y los acordes de la *Pieza III*. Aquí, los saltos entre sonoridades de baja y alta densidades son aún más abundantes que en las *Piezas I* y *II*, dentro de una evolución en extremo breve que genera a su paso hallazgos que no tienden a repetirse. La sensación de lance se acentúa aún más que antes en la obra, y en particular cuando se alcanza a escuchar completa por primera la sonoridad resultante de la escala menor armónica.

La alternancia de lo secuencial con lo vertical en la textura —melodía rápida y acordes con los que colisiona— es puesta en relieve por la red, que crea una



Figura 17: Schoenberg, *Seis piezas para piano*, fragmento inicial de la *Pieza V*.

estructura con forma de estrella en torno de la identidad 1 [12], punto desde el cual se desprenden casi todas las conexiones. El diseño que crea la red da la idea de una búsqueda cuyo proceso se basa en pescar aquí y allá nuevas sonoridades resultantes sin que éstas mantengan necesariamente una amplitud de vínculos, de no ser en el nivel N5 al centro de la red, donde se distingue el escaso vínculo entre las identidades 42 [1,1,3,3,4] y 44 [1,2,2,3,4]. La conectividad mediante saltos provoca el encuentro a mayor distancia que se haya dado hasta ahora en la obra al conectar la identidad 1 [12] con la 64 [1,1,1,2,2,2,3]. Esa misma franqueza en la conexión entre sonoridades resultantes apartadas conduce a considerar que las relaciones establecidas en las piezas anteriores forman ya parte de una memoria que facilita la creación de nuevos vínculos dentro del espacio de la escala menor armónica a la que la obra refiere con insistencia.

### 7.5. *Pieza V, melodía con acompañamiento*

El carácter ternario de la *Pieza I* reaparece aquí para dar al todo el aire de una suite moderna cuyo efecto de unidad formal beneficia la percepción del conjunto de materiales. El parentesco de búsqueda es amplio en ambas piezas al observar una vez más la contradicción que resulta de oponer armonías de distintos orígenes tonales, como los arpegios encontrados del inicio —Sib menor arriba y Do menor abajo— o al mantener la tendencia a conectar la identidad 1 [12] con sonoridades resultantes de mayor densidad. La diferencia que más resalta aquí respecto a la primera pieza es que el ritmo tiende a procurar homogeneidad a la articulación de las voces mediante la sincronía, un aspecto que a su vez vincula esta quinta pieza al carácter coral de la *Pieza III*.

La red que genera esta pieza presenta el mayor equilibrio en la distribución de sus distintas densidades verticales, en la conectividad entre identidades del mismo nivel y entre identidades distantes. Resurge aquí el motivo de tercera mayor —Ii 5 [4 8]— así como la extensa conectividad de la identidad 1 [12] con que inicia la pieza y que deviene otra vez un eje de la exploración. El conjunto de sonoridades resultantes se puede derivar nuevamente de la zona I, escala menor armónica, identidad 64 [1,1,1,2,2,2,3], a mínima distancia de una de las identidades de nivel N6, límite de mayor densidad. La pieza finaliza con dos sonoridades resultantes que aparecen en la obra por vez primera, la 56 [1,1,2,2,3,3], que precede a la identidad 55 [1,1,2,2,2,4], al final (f), ambas de nivel N6 y asociadas a la escala diatónica. La exploración de las dos escalas

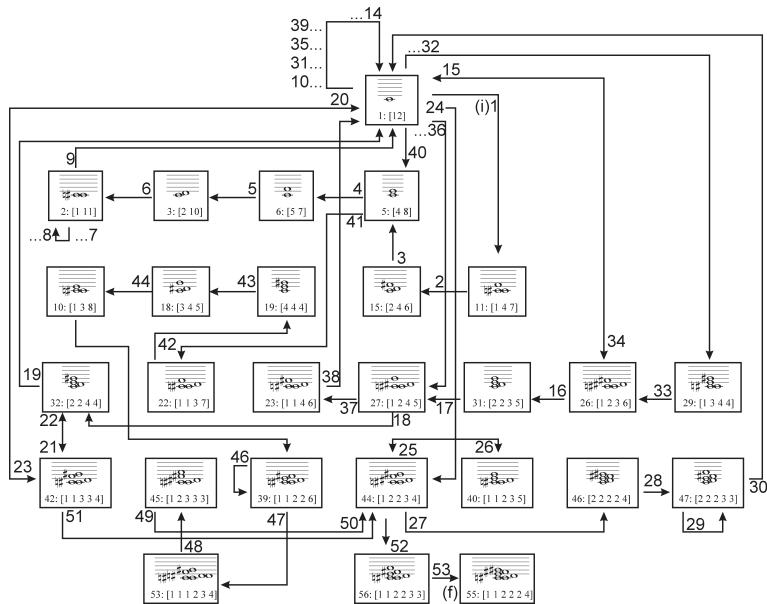


Figura 18: Schoenberg, *Pieza V*, red entre 27 identidades en un espacio continuo de niveles que abarca del  $N1$  al  $N6$ .

de referencia concluye con esta penúltima pieza, experiencia de búsqueda exhaustiva de los últimos vestigios del mundo tonal, lo que acaso permite al autor llevar a cabo una apertura al final de la obra para romper de manera definitiva con sus raíces.

### 7.6. *Pieza VI, planos armónicos*

La última de las *Six pieces* revela una transformación drástica del proceso de búsqueda, hasta ahora tendiente a recorrer el espacio según las fórmulas melodía-armonía o melodía-melodía. Avanzar de nivel parece en toda la obra una condición obligatoria para aproximarse a los confines de la tonalidad, nivel  $N7$  al que llega la *Pieza IV* al incluir la sonoridad resultante de la escala

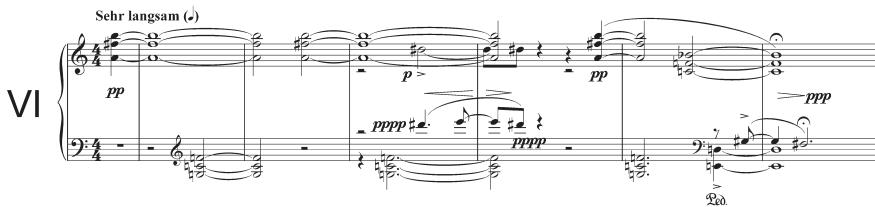


Figura 19: Schoenberg, *Seis piezas para piano*, fragmento inicial de la *Pieza VI*.

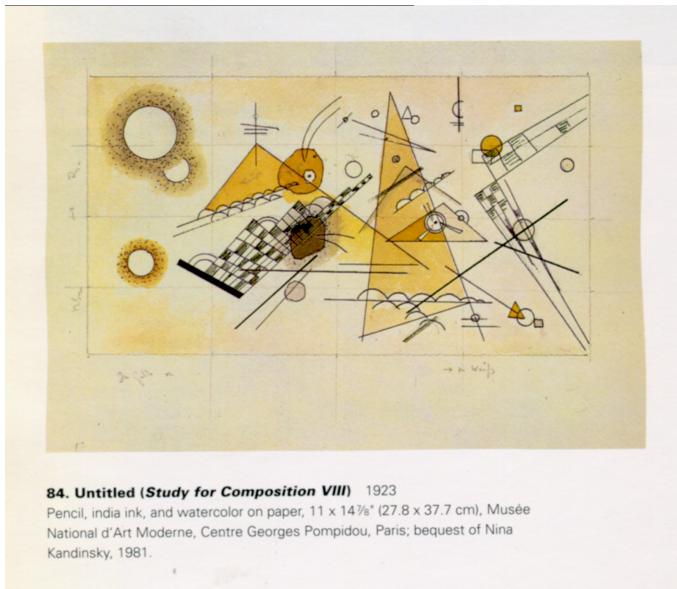


Figura 20: Vasily Kandinsky, *Estudio para Composición VIII* (1923), lápiz, tinta china y acuarela en papel, Museo Nacional de Arte Moderno, Centro Georges Pompidou, París, donación de Nina Kandinsky, 1981. Obsérvense el juego de superposición entre los triángulos al centro de la imagen, cuya percepción puede asimilarse a la audición de los planos en Schoenberg [3].

menor armónica. Las condiciones están dadas para que la *Pieza VI* rebase dicho umbral, aunque el autor no va a recurrir a las mismas texturas que antes para agenciarla la meta. La ambigüedad entre tonalidades o los cruces sorpresivos a los que acude previamente crearían demasiado disloque en los máximos de densidad que vislumbra; recursos como la poli-tonalidad no funcionan porque su exceso en densidad resulta borroso, indeterminación ajena al pensar de Schoenberg: se requiere de una estrategia inusual que conduzca al hallazgo de la textura adecuada. La partitura misma lo desvela al acoplar sonoridades resultantes como si fuesen planos que no dan la impresión de chocar sino de permitir la escucha autónoma de cada uno. La música se aleja de su lenguaje y se acerca, acaso también se adelanta, a la serie de composiciones de Vasily Kandinsky (1866-1944), cuyas figuras geométricas se traslanan sin encubrirse y crean una imagen cristalina.

La edificación textural schoenberguiana recurre para ello a una pasividad rítmica que contribuye a suspender la resonancia de las armonías y hacer que sus superficies transluzcan y se fusionen a otras armonías que penden en una secuencia que se oye con el sigilo de mínimas intensidades —*pp*, *ppp*, *pppp*. La nueva música deviene representación de una materia cuyo aislamiento la hace abstracta e impide que el oído pueda percibirla dentro del antiguo engranaje

de la función tonal. La autonomía de cada acorde nace de otro universo, un juego entre alturas de concepción similar a la del intervalo-objeto que despunta en la *Pieza III*, ahora como armonía-objeto cuya suspensión y traslape a otros acordes crea una libre mixtura de prismas sonoros. La armonía-objeto, por lo general tres alturas, parece evocar las figuras de Kandinsky con triángulos como el escaleno —La, Fa♯ Si— o el equilátero —Sol Do Fa, o Do Fa Sib—, a la vez que los pequeños trazos melódicos —compases 5 y 6—, como gotas que derraman el vaso, se inclinan hacia un nuevo límite.

El predominio radical de la verticalidad conduce al cabo de la *Pieza VI* al máximo de densidad en la obra, como confirma la síntesis novedosa de la red entre las identidades: dentro de un mínimo de 26 cortes verticales se denotan ocho escasas sonoridades resultantes; cinco de ellas —identidades 1 [12], 14 [2, 3, 7], 52 [1, 1, 1, 2, 2, 5], 53 [1, 1, 1, 2, 3, 4] y 64 [1, 1, 1, 2, 2, 2, 3]— aún dentro del ámbito de la escala menor armónica, evocada por segunda vez (Cf. *Pieza IV*); el resto proviene de una determinación en el avance y de una suavidad para lograrlo mediante mínimas distancias *d1* hacia tres sonoridades resultantes inexploradas: en el propio nivel N7, la identidad 60 [1, 1, 1, 1, 1, 2, 5], en el N8, la identidad 69 [1, 1, 1, 1, 1, 2, 2, 3], y en el N9, la de mayor densidad vertical, la identidad 73 [1, 1, 1, 1, 1, 1, 2, 2, 2] —apenas a 3 mínimas distancias del límite, la identidad 77 [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]. Las tres nuevas identidades, de máxima disonancia —5 y 6 semitonos—, sobrepasan al universo de las escalas diatónica y menor armónica. La afanosa exploración que emprende el autor logra un impulso final cuyo desenlace, la atonalidad, es el punto de partida de una búsqueda que le conduce diez años más tarde a adoptar el planteamiento formal de la serie dodecafónica.

## 7.7. Síntesis de la búsqueda armónica

La brevedad de las *Seis piezas para piano* contrasta con la abundancia de sus sonoridades resultantes, que abarcan de forma continua los niveles de densidad N1 al N9. La innovación de Schoenberg reside en el ensayo de fusionar universos distintos, aunque no obligadamente disociados de observar su proyección dentro del continuo de identidades del potencial combinatorio de la escala D12:

La exploración completa del opus 19 denota la existencia de dos espacios distintos:

- la forma del espacio de búsqueda abarca casi todo el espacio cubierto por los sistemas sincrónico-armónicos; las agrupaciones que contienen mayor número de semitonos se dirigen como puntas hacia la parte superior de los niveles N3, N4, N5 y N7;
- otra punta se prolonga hacia el extremo derecho de la forma y alcanza los niveles N8 —zona K: línea punteada— y N9 —zona L: línea delgada—, mismos que integran las seis identidades de mayor nivel.

Schoenberg tiende en cada una de las seis piezas a crear una oscilación entre máximos y mínimos en el número de identidades, que a su vez complementa

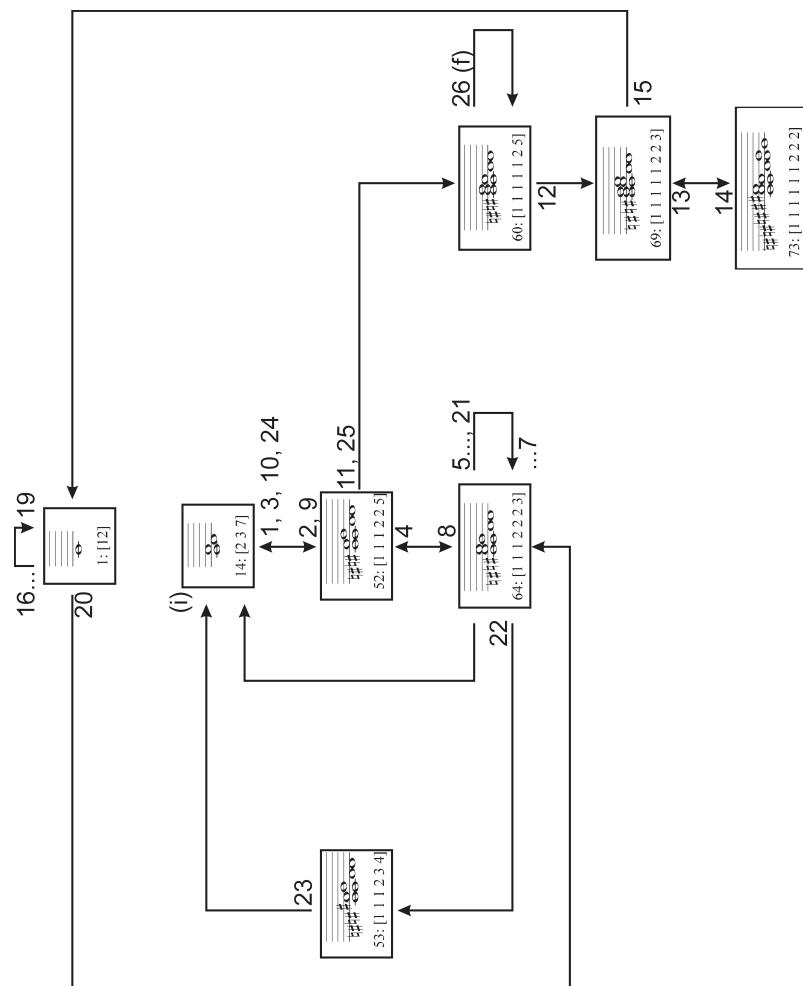


Figura 21: Schoenberg, *Pieza VI*, red entre 8 identidades en un espacio discontinuo de niveles que abarca el  $N1$ ,  $N3$ , y del  $N6$  al  $N9$ .

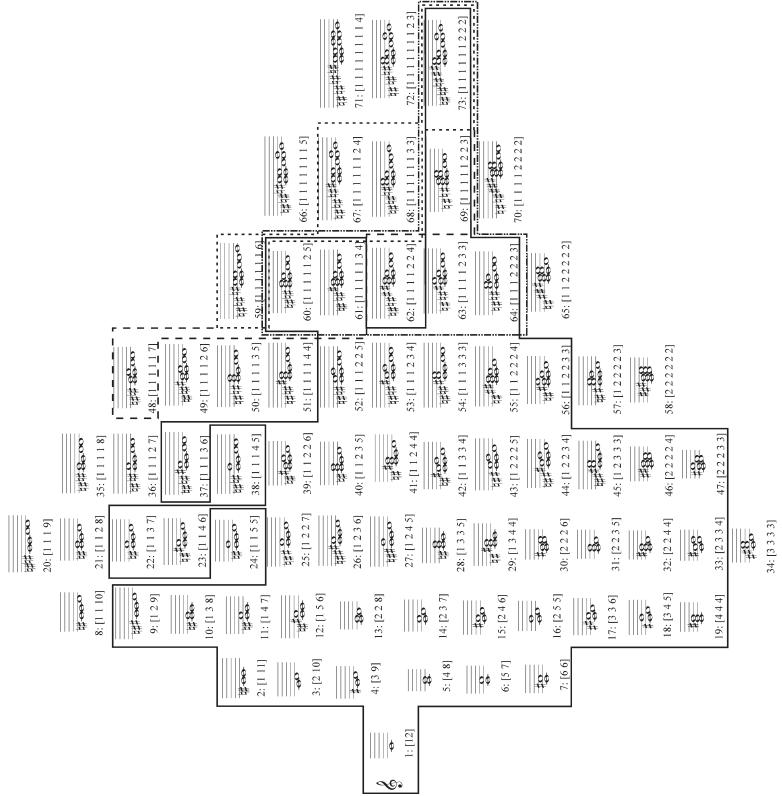


Figura 22: Schoenberg, *Seis piezas para piano*, proyección del espacio de búsqueda dentro del potencial combinatorio de la escala D12, 50 identidades distribuidas de forma continua en los niveles  $N1$  al  $N9$ . La línea continua enmarca el espacio de búsqueda y la línea discontinua indica las identidades que pueden vincularse al límite de la obra, la identidad 73 [1, 1, 1, 1, 1, 2, 2, 2].

con la alternancia, respectivamente, entre texturas tradicionales e innovadoras, como se evidencia mediante su comparación por pares:

- I, polifonía libre, 34 identidades - II, intervalo-objeto, 9 identidades,
- III, coral, 16 identidades - IV, alternancia secuencial-vertical, 10 identidades,
- V, melodía con acompañamiento, 27 identidades - VI, planos armónicos, 8 identidades.

La secuencia parece calcular el zigzag de piezas noes a pares<sup>9</sup>, de una conectiva que va de mínimas a máximas distancias —factor que influye en que el resultado se perciba por momentos con cierta estridencia—, de sonoridades resultantes conocidas a otras de nueva adopción —con frecuencia disimuladas en el discurso, a excepción del declarativo final de la obra. Singular creación musical, induce a captar su dualidad, sello que opone consonancia y disonancia, tradición y novedad, tonalidad y atonalidad, lo que invita a ubicar las *Seis piezas para piano* en la intersección de tres espacios musicales, consonante (A.), disonante (B.) y atonal extendido (C.), convivencia que resulta de una rica amalgama de conocimientos, originalidad y creatividad.

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<sup>9</sup>Dicha idea recuerda en cierto modo la relación contrastada que crean las sinfonías noes de Beethoven, más aventuradas que las pares, de carácter más clásico.

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# Preludio a tres partes y $\phi$

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## Resumen

En el presente documento muestro la aplicación de un sencillo algoritmo propuesto por Charles Madden [7] en la composición de una obra musical para piano, un preludio a tres partes. Inicio con la descripción del algoritmo mencionado que incluye un breve análisis del número  $\phi$ , posteriormente describo la composición detallando el cambio estructural de la melodía y la verificación de la relación  $\phi$ , de acuerdo con Madden.

## 1. Introducción

La denominada razón dorada ha sido una expresión matemática que cautiva los intereses de investigación y aplicación desde hace mucho tiempo en el quehacer matemático y musical. También es muy conocida su relación con la naturaleza y su aplicación en la pintura y arquitectura. En música, el genio de Eisenach, Juan Sebastián Bach, utilizó la relación en 6 preludios de su obra maestra *El clavecín bien temperado*: del libro 1 números 9, 13, 17, 21; del libro 2 números 19 y 21. Después de él, muchos hemos seguido deliberadamente o no con la idea de aplicar en las composiciones la Sección Dorada (SD):

1. Haydn, en las *Sinfonías* números 90 y 98; *Cuartetos de cuerdas* Op. 9 Nos. 2 y 6, Op. 17 No. 2, Op. 42, Op. 50 No. 5, Op. 55 No. 3, Op. 64 Nos. 2 y 4, Op. 74 No. 1, Op. 76 No. 1; la *Sonata* No. 25.
2. Mozart, en sus *Sinfonías* K18, K128, K183 y K551; *Cuartetos de cuerdas* K159, K168, K169; *Sonatas* para el piano: K279, K333, K545.
3. Beethoven, *Cuarteto de cuerdas* Op. 18 No. 4.
4. Chopin, en los *Preludios* Op. 8, 17, 20 y 23; *Nocturnos* Op. 15 No. 2, Op. 48 No. 1, Op. 55 No. 2, Op. 62 No. 1, Op. 72 No. 1;
5. Brahms, Op. 39 Nos. 6 y 12.
6. Bartók, en sus obras: *Música para cuerdas* III, *Cuarteto de cuerdas* 5, I y II, *Allegro Bárbaro*, *Mikrokosmos* 140, *Divertimento* I, *Contrastes* I.
7. Schönberg, en sus composiciones: *Pieza para piano* Op. 33a, *Quinteto para viento*, (cuarto movimiento), *Cuarteto para cuerdas* No. 2 (primer movimiento) y No. 3 (primer movimiento).
8. Stockhausen, en *Adieu, Klavierstück* No. 9.
9. Debussy, en sus obras: *Spleen*, *Reflets*, *L'isle joyeuse*, *Nuits d'étoiles*, *Mandoline*, Cuarteto de cuerdas, *La Mer* III.

Hay muchos autores más; para mayor información, consúltese la obra de Madden [7]. Sin embargo, la cuantificación de ello no se había estructurado adecuadamente. Desde el siglo XIX, los matemáticos que gustaban de la música se dieron a la tarea de hacerlo, y en nuestra época, quien ha recopilado esos algoritmos numéricos relacionados con SD, es Charles Madden. He seguido su trabajo, y el algoritmo que propone lo aplico para componer una obra musical, un preludio a tres partes, en 2007. Dicha composición la intenté ajustar a la distribución de SD en la primera y segunda parte y  $SD^2$  en la segunda parte y deliberadamente no la apliqué a la tercera. Doy a conocer el resultado del análisis de dos de ellos.

## 2. El método propuesto por Charles Madden

El hecho de multiplicar o dividir consecutivamente una secuencia de proporciones  $\phi$  por uno o varios de sus componentes, es en realidad una operación que consiste en modificar también proporcionalmente el valor inicial, es decir, un valor autosimilar. En el Cuadro 1 se puede observar la relación proporcional entre  $\phi$  y la Sección Dorada (SD). Se pueden apreciar valores como  $\phi^2$ ,  $\phi$ , 1 y  $\frac{1}{\phi}$ ,

Potencia de $\phi$	Potencia de SD	Valor Numérico
$\frac{1}{\phi^4}$	$SD^4$	0.146
$\frac{1}{\phi^3}$	$SD^3$	0.236
$\frac{1}{\phi^2}$	$SD^2$	0.382
$\frac{1}{\phi}$	SD	0.618
$\frac{1}{\phi^0}$	$SD^0$	1.000
$\phi$	$SD^{-1}$	1.618
$\phi^2$	$SD^{-2}$	2.618
$\phi^3$	$SD^{-3}$	4.236
$\phi^4$	$SD^{-4}$	6.854

Table 1: Relación entre  $\phi$  y la Sección Dorada (S. D.).

que también pueden ser descritos en términos de la división por  $\phi$ :  $\phi$ , 1,  $\frac{1}{\phi}$  y  $\frac{1}{\phi^2}$  y también en términos de la multiplicación por  $\phi$ :  $\phi^3$ ,  $\phi^2$ ,  $\phi$ , y 1. De éste modo una composición musical está representada por la unidad (1), y sus componentes estructurales agrupados en determinado número de compases deben ser porciones de la unidad, en términos de:  $\frac{1}{\phi}$ ,  $\frac{1}{\phi^2}$ ,  $\frac{1}{\phi^3}$ , etc., lo que conduce a la idea de la Sección Dorada: SD,  $SD^2$ ,  $SD^3$ ,  $SD^4$ , etc. En los análisis de obras musicales que realiza Madden, no incluye valores mayores que la unidad, pero a los respectivos valores menores que la unidad se hace referencia como la proporción dorada en las obras musicales. Es importante recordar que  $\phi$  tiene varios nombres, como la “razón dorada”, “media geométrica dorada”, “sección dorada”, “proporción dorada” y por supuesto “divina proporción”, lo comentó porque debemos tener

presente la palabra “proporción” y así distinguir la diferencia entre una razón y una proporción. Por ejemplo,  $A : B$  y  $B : A$  son razones; es decir son partes que se relacionan la una con la otra. Pero  $A : (A + B)$  y  $B : (A + B)$  son proporciones pues relacionan las partes con un todo. Las proporciones  $B : A$  y  $A : (A + B)$  son iguales e intercambiables sí y sólo si ambos son exactamente  $\phi$ . De acuerdo con ello, es importante convertir las razones a proporciones al analizar partituras musicales, antes de calcular cualquier porcentaje y suponer, erróneamente, que se ha encontrado alguna proporción dorada en ellas. Por ejemplo, si  $A = 3$  y  $B = 2$ , su razón se puede expresar de la forma  $A : B$ , o  $3 : 2$ . Sin embargo, para poder expresar como la proporción de una composición musical, es necesario tener la forma:

$$A : (A + B) = 3/(3 + 2) = 3/5 = 0.6.$$

También es posible expresar:

$$B : (A + B) = 2/(3 + 2) = 2/5 = 0.4.$$

Existe una gran diferencia entre  $3/2 = 1.5$  y  $2/3 = 0.667$ . Las proporciones en el contexto del análisis de composiciones musicales, deben ser complementarias; es decir, añadiendo las proporciones a la unidad. Esto es posible al dividir cada numerador,  $A$  o  $B$ , por su suma, formando un mínimo común denominador. Es justamente lo que se propone,  $A/(A + B)$  y  $B/(A + B)$ , para todas las proporciones encontradas en las partituras de las composiciones musicales, (un ejemplo puede ser la fracción o el porcentaje de ellas consideradas como un todo). Las razones de los números de Fibonacci se transforman a proporciones. De ésta forma, la razón  $3 : 5$  se transforma en  $3/8 = 0.375$ . En concreto, la notación utilizada para representar la sección dorada SD es: NF, para cualquier número de Fibonacci; NL, para cualquier número de Lucas y NG representa a un número de Fibonacci generalizado. Éstos tres términos (NF, NL, NG) no necesariamente representan valores numéricos, como sí lo hacen SD,  $SD^2$  sino que más bien identifican secuencias musicales a las que se asocia un número particular, tales secuencias musicales se encuentran como múltiplos de ellos, como 2FN, en donde todos los números de la secuencia se multiplican por 2. Como es conocido, también existe la cualidad aditiva de las proporciones  $\phi$ , permitiendo una mejor comprensión de las relaciones  $\phi$ . Muchas combinaciones aditivas son más simples que las multiplicativas, en cuanto a forma. Estas relaciones deben tenerse muy presentes pues con frecuencia son sumamente necesarias al analizar

composiciones musicales. Los equivalentes aditivos de la proporción  $\phi$  son:

$$\frac{1}{\phi} = \phi - 1, \quad (1)$$

$$\frac{1}{\phi^2} = 1 - \frac{1}{\phi} = 2 - \phi, \quad (2)$$

$$\frac{1}{\phi^3} = \frac{1}{\phi^2\phi} = \frac{1}{\phi^2 + \phi} = \frac{1}{2\phi + 1}, \quad (3)$$

$$\phi = 1 + \frac{1}{\phi}, \quad (4)$$

$$\phi^2 = \phi + 1, \quad (5)$$

$$\phi^3 = \phi^2\phi = \phi^2 + \phi = 2\phi + 1. \quad (6)$$

### 3. Los niveles de seguridad

Como producto de los trabajos de investigación en éste tema realizados desde el siglo XIX, en los que los niveles de tolerancia eran bastante amplios, ha sido necesario implementar un estándar estricto, que permita identificar qué tan cerca se fija un NF con respecto a SD. Se puede apreciar que cada proporción Fibonacci  $\frac{F_n}{F_{n+1}}$  se aproxima a SD, aunque en dos casos como  $\frac{1}{1}$  y  $\frac{1}{2}$ , no es así.

Por ejemplo, las proporciones  $\frac{3}{5} = 0.6$ ,  $\frac{5}{8} = 0.625$  y  $\frac{2}{3} = 0.667$ , podrían parecer cercanas a SD en obras musicales pequeñas, pero en composiciones musicales muy extensas, digamos de 1000 compases (aunque es poco frecuente), la diferencia entre SD (0.618) y  $\frac{5}{8} = 0.6$ , estarían implicados 18 compases. De éste modo, al dividir el valor de la diferencia con el valor correcto, en éste ejemplo  $\frac{0.018}{0.618} = 0.0291$ , es decir se obtiene aproximadamente el 3%. Con un grado de pérdida de exactitud como el anterior, obliga a revisar cuidadosamente los cálculos a realizar cuando se analizan composiciones musicales. En investigaciones anteriores, un error del 8% (0.667 de 0.618) no hacía percibir una pérdida del concepto de  $\phi$ , por ello las tolerancias en la actualidad deben ser más rigurosas. Con ayuda de las computadoras modernas, estudios realizados han verificado las relaciones fib y  $\phi$  en las estructuras musicales con un alto grado de precisión, como en los trabajos de Dunlap [5], Prusinkiewicz [9], quienes han demostrado que los modelos de estas estructuras requieren una precisión del orden del 0.1%, así que en las investigaciones realizadas posteriormente, las tolerancias se acercan a éste último valor, permitiendo una varianza mínima, 0.617 a 0.619 (el valor de comparación es 0.618); en reportes anteriores sobre el tema, no se tenía una metodología consistente, y debido a ello muchos analistas no tomaban en cuenta las introducciones y las codas de las composiciones musicales, para así adaptar el resultado del análisis con la SD. En su报告 Madden toma en cuenta lo anterior, pues así es posible percibirse qué autor musical ha utilizado conscientemente NF o SD y quién no, sobre todo en autores contemporáneos que usan fib y  $\phi$  en sus obras.

## 4. Procedimiento de análisis de composiciones musicales

Para encontrar las proporciones en una composición musical, el autor referido utiliza el siguiente procedimiento:

1. Encontrar el número total de compases de la obra.
2. Multiplicar por el valor de SD o  $SD^2$ .
3. Redondear el valor del resultado al entero más próximo.
4. Distinguir los compases o unidades que corresponden a SD o  $SD^2$ .
5. Determinar si existen en la obra los puntos armónicos o las secciones estructurales.
6. Subdividir la composición u obra musical para buscar otras relaciones  $\phi$ .
7. Asegurarse de que la extensión de los compases relacionados con SD y  $SD^2$  termina una sección o frase y no la inicia.

La siguiente tabla muestra los diez primeros múltiplos de los valores 0.618 y 0.381924. El procedimiento se ilustra con el siguiente ejemplo, en el que se toma

Factor	GS	Resultado	$GS^2$	Resultado
0	0.618	0	0.381924	0
1	0.618	0.618	0.381924	0.381924
2	0.618	1.236	0.381924	0.763848
3	0.618	1.854	0.381924	1.145772
4	0.618	2.472	0.381924	1.527696
5	0.618	3.090	0.381924	1.909620
6	0.618	3.708	0.381924	2.291544
7	0.618	4.326	0.381924	2.673468
8	0.618	4.326	0.381924	3.055392
9	0.618	5.562	0.381924	3.437316

Table 2: Múltiplos de  $\phi$  y  $\phi^2$ .

el valor de 78 como el número total de compases de una partitura cualquiera:

1.  $(0.618)(78) = 48.204$
2. Redondeando al entero más próximo: 48.
3.  $(0.381924)(78) = 29.790072$
4. Redondeando al entero más próximo: 30.

Entonces el valor de SD es 48 y  $SD^2$  es 30. Sin embargo, lo importante es notar la existencia de un punto estructural en un compás o unidad melódica para que exista una proporción  $\phi$ , ya que de lo contrario, ésta no sería posible. En éste punto, el autor comenta en su libro que es muy difícil hacer notar a sus lectores si realmente existe el punto estructural en alguna melodía, pero que es importante poner mucha atención al analizar una partitura, observar con

cuidado si existe una unidad melódica, es decir, una idea musical, etc., además observar una pausa estructural, un cambio de textura o en la dinámica de la composición y no sólo analizar los cambios en la estructura armónica, sino analizar tanto la melodía como la armonía. Es importante notar que si la armonía no coincide con la SD, simplemente puede significar que el compositor no estaba utilizando conscientemente la SD con propósitos estructurales. El siguiente paso es observar cuidadosamente el punto estructural y revisar si coincide con un NF, NL o SD. Si no es así, el compositor no estaba pensando crear su composición en éstos términos. Concluyendo, la verdadera proporción se localiza aplicando el porcentaje calculado al dividir la longitud de la sección musical principal de la composición entre la longitud de toda la composición y establecer una aproximación a NF o SD. Si se ha observado una situación como la anterior, se debe intentar buscar subgrupos que guarden una relación así, pues muchos compositores, como Karlheinz Stockhausen, utilizan NF o SD de ese modo. Otros, como Varése o Schönberg utilizan la SD solamente en estructuras grandes y muchos otros la han incluido en sus composiciones intuitivamente. Muchas disertaciones se han utilizado como valiosas fuentes de información sobre el uso de fib y  $\phi$  en la música, incluyendo aquellas hechas por Pascoe [8], Rogers [10], Rothwell [11] y Dorfman [3], Adkins [1], Camp [2], Demaree [4], Hoffman [6] y Sidorowicz [12], Madden las incluyó en el capítulo correspondiente de tal modo que aporta un valioso resumen de éstos trabajos previos además de contribuir con otra verificación de ellos, y determinar la profundidad con que  $\phi$  está presente, y establecer la forma de que los resultados sean visibles. De esa manera, todas las sonatas para piano, los cuartetos de cuerda y las sinfonías de Haydn, Mozart y Beethoven se analizaron, concluyendo, después de ello, qué tanto los compositores de la época clásica conocieron y aplicaron  $\phi$  y fib en sus obras musicales. Aquí es prudente aclarar la errónea idea que se ha tenido acerca de de que el primer movimiento de la Quinta Sinfonía de Beethoven estuviese dividida exactamente en términos de la secuencia de Fibonacci. De todos ellos, la disertación [12] es de especial interés. En dicho documento, Sidorowicz analiza 25 sonatas de Beethoven tomando en cuenta todos los compases y puntuizando los cocientes correspondientes para determinar si SD o SD<sup>2</sup> están presentes. Acertadamente, toma en cuenta la exposición, el desarrollo, la recapitulación y la coda, como componentes de la forma sonata, puntuizando el número de compases de cada uno de ellos.

## 5. El preludio

Un preludio, en música, es una pieza destinada a tocarse como introducción, por ejemplo en una ceremonia litúrgica o inclusive en otra obra musical, como la fuga o la suite. Ésta forma de usarse prevalece desde sus comienzos, en 1450, pero perdió el carácter en el siglo XIX, pues Chopin, Scriabin, y Debussy, entre otros, emplearon la palabra como uno de muchos títulos imprecisos de piezas. Con pocas excepciones, el preludio se ha limitado a solos de música instrumental, especialmente con el teclado y el laúd. Durante el siglo XV era una pieza corta,

de diez a veinte compases, de estilo libre. A partir de 1650 los compositores lo empezaron a combinar como introducción a suites, imponiéndose la moda de ser en estilo libre e improvisatorio, aunque en J. S. Bach son movimientos completos incluidos en el concierto “grosso”, las oberturas, las tocadas o las sinfonías (por ejemplo los preludios y fugas). El siglo XIX, se encuentra representado por los de Chopin, y sus sucesores, ya mencionados más arriba, además de Rachmaninov. Son piezas esencialmente de carácter pianístico.

### 5.1. Preludio No. 7 en Re Bemol Mayor

La obra musical es una composición, en tres partes, que realicé en 2007 y es el séptimo preludio que junto con otros cinco y tres valses forman parte, hasta la fecha, de las Composiciones para Piano que iniciara en el año de 1973. Su título corresponde, como ya mencioné, a una estructura musical libre, aunque debo hacer notar que es muy extensa para clasificarse estrictamente en el género preludio. Sin embargo, al tratarla de ubicar en otro como el de la sonatina, observé que su diseño no tiene los elementos característicos en ella, ni de la sonata (introducción, exposición, desarrollo, conclusión y coda). Si bien la colección de sonatinas de Muzio Clementi son cortas y en forma breve, contienen dichos elementos, por lo que la obra no se ajustaría del todo. La primera parte a un compás de  $\frac{4}{4}$  en tono de Re Bemol Mayor, en *vivace*, se ejecuta con una serie de arpegios que van recorriendo la gama del conjunto de notas del tono correspondiente, la melodía corre a cargo de la mano izquierda sobre la octava de Do 4, brincando para continuarla hasta la gama de Do 7. Un cambio de idea musical surge en tono de Mi Bemol Mayor para terminar la parte primera regresando al tono original. Consta de 53 compases en total. Aplicando el algoritmo:

$$53 \times 0.618 = 32.754,$$

es decir, el compás 33. En éste caso, el término de la frase musical ocurre en el compás 33, como se puede apreciar en la Figura 1.

Para verificar el dato, de acuerdo con el algoritmo:

$$\frac{33}{53} = 0.662$$

y estableciendo la proporción:

$$\frac{0.618}{100} = \frac{0.622}{X}$$

resultando 100.6472, es decir, un margen de error en la composición respecto de la SD de 0.6472%. Ahora, para éste mismo movimiento, pero con relación a  $SD^2$ :

$$53 \times 0.318 = 20.24,$$

es decir, el compás 20. Al observar la Figura 2, se puede el lector percibir que no existe un cambio estructural en la melodía ni en la armonía, ya que no pude ajustarla pues perdía totalmente la estructura original de la composición.

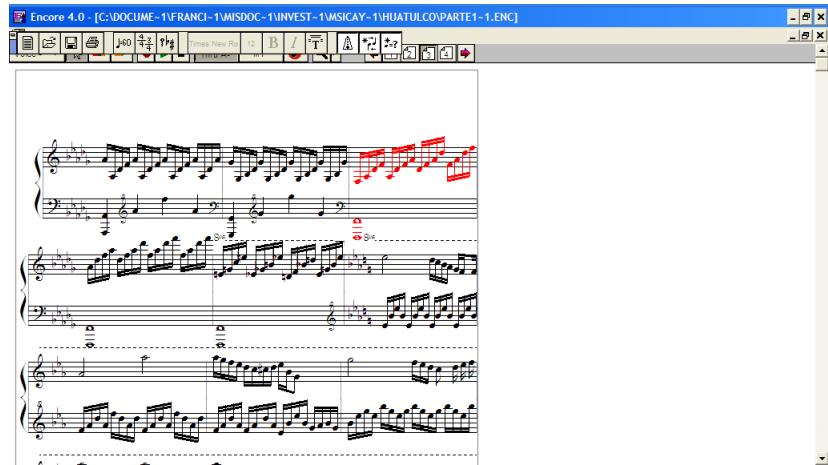


Figura 1: El compás 33, (con notas en color rojo), muestra el fin de la frase musical, en tono de Re bemol Mayor, iniciando otra idea musical en mi bemol mayor, en el compás 36.

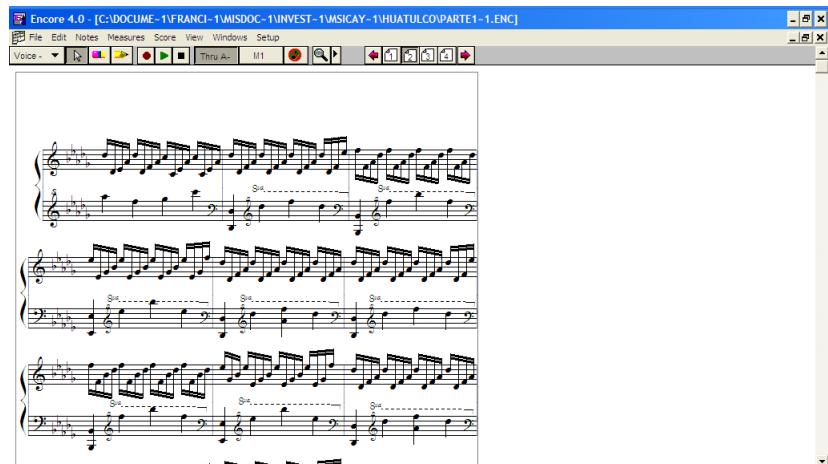


Figura 2: SD<sup>2</sup>. Pese a varios intentos no logré consolidar la aplicación del algoritmo de Madden en ésta parte del preludio, pues el vigésimo compás (aquí el número 5, segundo renglón, al centro), pese a que termina una frase, no cambia estructuralmente de idea musical.

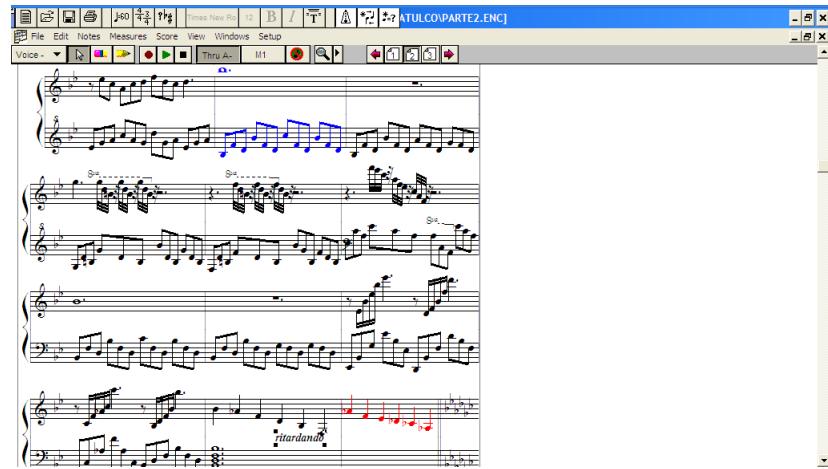


Figura 3: Se observa el compás número 27 (notas en color rojo) con el que termina la idea musical inicial y cambia de Si bemol mayor a Sol bemol mayor, y también de idea.

La segunda parte, de ritmo ternario a  $\frac{12}{8}$  con tiempo de *allegretto*, en tono de Si Bemol Mayor, donde la melodía está a cargo de la mano derecha sobre la octava de Do 6 y Do 7 de carácter suave y más lento que en la primera parte. Al terminar la idea original de ésta parte, se cambia de tono a Sol Bemol Mayor, con el mismo ritmo ternario, y una frase musical distinta a la primera, suave, dulce, melodiosa, en *andantino cantabile*. La segunda parte completa consta de 44 compases, de manera que:

$$44 \times 0.618 = 27.192,$$

situándose en el compás 27.

Verificando:

$$\frac{27}{44} = 0.61363$$

y estableciendo la proporción:

$$\frac{0.618}{100} = \frac{0.613}{X}$$

obteniendo el valor de 99.29 de modo que la diferencia en porcentaje es de 0.71. Ahora, para el caso de SD<sup>2</sup>:

$$0.3819 \times 44 = 16.60,$$

es decir, el compás 17. La figura 4 muestra el cambio de idea musical.

La tercera parte del preludio no la realicé con base en la SD intencionalmente, para dejar que mi imaginación musical no se ajustara a esta métrica, y permitir que prevaleciera la inspiración como tal y para contrastar los dos estilos de

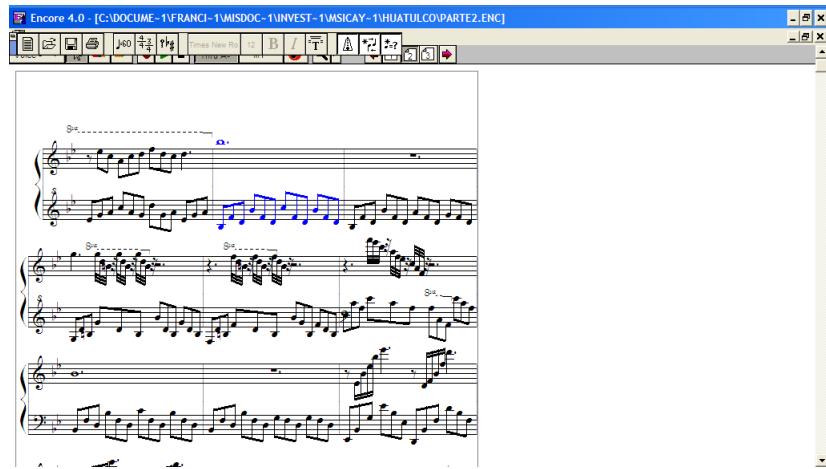


Figura 4: El compás 17 (con notas en color azul), de la segunda parte del Preludio. Termina la frase musical.

generar una composición. inicia en Do menor en agitato assai, a cuatro por cuatro, fluyendo el tema con base en la tónica, la subdominante y la séptima sensible, para cambiar de tono a do sostenido menor y repitiendo la idea musical anterior, para finalmente terminar con una coda que retoma la idea musical y el tono de Re bemol mayor del primer movimiento.

## 6. Conclusión

La técnica de conteo propuesta por Madden, quien recopila los trabajos hechos por investigadores anteriores en este tema, permite identificar de forma confiable qué tanto una obra musical se adapta a SD o  $SD^2$ , inclusive. El propósito se encamina a relacionar las composiciones musicales basadas en las formas establecidas con la maravillosa relación de  $\phi$  y fib. Un aspecto interesante de ésta participación es haber verificado la utilidad de la propuesta en una obra musical creada en 2007, que titulé *Preludio a Tres Partes*, y que intenté ajustarla con  $\phi$  y  $\phi^2$ ; logré en la primera parte aplicar  $\phi$  pero no  $\phi^2$ ; en la segunda parte conseguí aplicar ambos, con muy pequeña desviación. La tercera parte deliberadamente omití la relación  $\phi$  y  $\phi^2$  a manera de reflexión entre seguir un modelo numérico y no seguirlo al componer una obra musical, dejando fluir mi creatividad y no limitarla al seguir un algoritmo propuesto. Queda pendiente, sin embargo, continuar con este tipo de trabajos como una forma de acercamiento al conocimiento humano en dos de sus ramas, la música y las matemáticas contando con otra forma de percibir esta relación tan especial, a través del sentido del oído en vez del sentido de la vista, como a través de la pintura o la arquitectura.

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**Book presentation:**  
**Una introducción a la Teoría de Grupos**  
**con aplicaciones en la Teoría**  
**Matemática de la Música**

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Mathematics is one of the Fine Arts —the purest of them—, who has the gift of being the most accurate and the accuracy of Sciences.

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E. Lluis-Puebla

This book is in Spanish, and it will be translated to English in the near future. Unlike the usual presentations of books in which the audience is expected to buy the edition from the publisher (so he can recover his huge investment), or from the author (who made his own edition and dispose of thousands of copies placed in his living room), in this occasion you have no need to do neither one nor the other. Instead, we give you a free copy of the book, as we do with anyone around the world that simply connects to the Web site of the Sociedad Matemática Mexicana at [www.smm.org.mx](http://www.smm.org.mx).

A friend of mine —who died a little more than two years ago and worked for a publishing company— once told me (still in the last century): “See Emilio, we buy paper cheaply and resell it more expensive. We are not interested in the black spots on the paper. A ‘good book’ is one that sells a lot of paper. Period.”

This book is published online and with open access (i.e. free) through the Internet, with the option of acquiring the work on CD or hard copy upon request. Later the costumer can print and leather-bound it in the skin of whatever animal he wants (crab, turtle, crocodile, lizard, etc.) which, of course, he should pay. The paper version has a cost as well as the CD version. The electronic version of the Sociedad Matemática Mexicana is free and can be copied to your computer freely. It is on the shelves of the world’s largest library, available to anyone interested in reading it, even if he has no money.

For us, mathematicians, what matters are those black spots, which need not to be embodied in an obsolete medium that destroys trees.

It turns out that, when I visited many universities in both Mexico and Latin America in particular, the libraries of some of them were smaller than mine. Also when I had an exhibition of my own books, I found they were already used as a text in many universities and those that had the paper version had to photocopy them because of the expensive and poor distribution of mathematical books published in paper.

Affordability is relative to the incomes of people in this or that country. Hence, when I was in charge of the presidency of the SMM from 2000 to 2002 I created the Electronic Publications of the SMM, providing the mathematical community in Latin America and the world a free access service. All electronic publications are subject to rigorous peer refereed as all of the SMM publications.

So we, the authors of this textbook, decided to include it within the Electronic Publications of the SMM with the aim to lead by example and show (as mathematicians and me as Executive Editor) confidence in this type of publication.

The Electronic Publications of the SMM constitute a library of free access to all the mathematical community in the country and the world. It consists of three series: Texts, Memoirs and Popularization books.

Texts series consists of two types of books: new textbooks written specifically for this media and other books that have been used for generations and which are now out of print.

It should be noted that electronic publications of the SMM incur no cost to our society. They are self-sufficient and provide a service to the community (particularly of mathematicians) from around the world. I want to invite you, as Executive Director of it, to publish in this splendid media.

The success of group theory is impressive and extraordinary. It is perhaps the most powerful and influential branch of the whole mathematics. It influences almost all scientific disciplines, arts (including Music) and Mathematics itself in a fundamental way.

The concept of structure and those related to it, such as isomorphism, play an important role in current mathematics. This text is based on the book “Group Theory: a first course” (in Spanish) written by myself, published in this series and which contains the material for the course on the subject I taught in the Faculty of Sciences of UNAM coupled with optional introductory material to a basic course in the Mathematical Theory of Music.

This text follows the approach of other of my texts on Linear Algebra and Homological Algebra. I chose a modern presentation which introduces the language of commutative diagrams and universal properties, as required in current mathematics and in Physics and Computer Science, among other disciplines.

The text consists of four chapters. Each section contains a number of problems that are solved creatively using the exposed material, and constitute an essential part of the text. They also have the purpose of allowing the student to write mathematics. Throughout the first three chapters we included interesting representative (and not numbered) examples of applications of groups used in Mathematical Music Theory for students who already have some knowledge of Music Theory.

Chapter IV describes in detail some applications of Group Theory to Music Theory. It explains some basic aspects of the Mathematical Music Theory and, during this process, it is intended to provide elements for readers of diverse backgrounds in both Mathematics and Music. See Mariana's comments in this volume for further details.

I may tell you that my original idea was to write this book, but it would take too much time, as it really did. So in 2006 I decided to publish just the part of Group Theory.

It took three more years to complete the project, thanks to my students, Mariana Montiel Hernández (who had worked in the U.S. for many years and wrote her part jointly with her masters student Janine du Plessis) and Octavio Alberto Agustín Aquino (to whom I thank his hard work, wisdom and good taste for the publication of this text.) They all collaborated on this new version.



# **Presentación del libro:**

## **Una introducción a la Teoría de Grupos**

### **con aplicaciones en la Teoría**

### **Matemática de la Música**

*Mariana Montiel  
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En el capítulo 4 radica mucho de lo novedoso de este texto. La idea del capítulo 4 surgió de la tesis de maestría Janine du Plessis, bajo mi dirección, en la Universidad Estatal de Georgia, que se titula “Grupos de transformación y dualidad en el análisis de las estructuras musicales.”

A sabiendas que el primer contacto de un lector potencial sería con el resumen (*abstract*) de su tesis, éste se desarrolló con el propósito de:

1. Dar a entender al posible lector que hay toda una manera alternativa de relacionar la Matemática y la Música. En otras palabras: que, tradicionalmente, el estudio de los intervalos entre tonos se hacía usando las razones de frecuencia de las potencias de enteros pequeños. Parafraseando a Gerald Balzano, en su bien conocido e importante artículo de 1980, “The group-theoretic description of 12-fold and microtonal pitch systems”, la Teoría Matemática de la Música moderna ofrece una manera independiente de entender el sistema de tonos, considerando los intervalos como transformaciones.
2. Aprovechar el surgimiento histórico de las estructuras algebraicas en la musicología y, en el espíritu de la Teoría Transformacional, concentrarse en las operaciones que forman grupos matemáticos.
3. Explorar y desarrollar aspectos de la teoría neo-riemanniana, en particular los grupos T/I y PLR como duales (idea surgida del artículo de Crans, Fiore y Satyendra [3]).
4. Presentar las herramientas teóricas de la Teoría de Grupos para el análisis musical y, sobre todo, aportar pruebas detalladas de muchas afirmaciones que son propuestas y usadas, pero rara vez demostradas.

Era este último punto el que detonó la idea de aumentar y realzar el libro de texto de Teoría de Grupos del Dr. Lluis Puebla, para incluir ejemplos de aplicaciones musicales en los primeros tres capítulos (una idea que él había estado gestando hacia varios años) así como un cuarto capítulo. En este cuarto capítulo, de una forma suave y “amigable”, se explicarían algunos aspectos básicos de la Teoría Matemática de la Música y, en el proceso, se darían elementos a lectores de diversos antecedentes, tanto en la Matemática como en la Música.

Por este motivo, los ejemplos del cuarto capítulo siguen de algunos de los aspectos teóricos sobresalientes de los capítulos previos; los objetos y términos musicales son introducidos conforme se vayan necesitando para que un lector sin formación musical pueda entender la esencia de cómo la Teoría de Grupos es empleada para explicar ciertas relaciones musicales ya establecidas. Asimismo, para el lector con conocimiento de la Teoría Musical, este capítulo provee elementos concretos, así como motivación, para comenzar a comprender la Teoría de Grupos.

En términos de antecedentes, podríamos mencionar los textos de Análisis Postonal, por ejemplo, usados en los Departamentos de Música en que, como en toda aplicación, la Teoría de Grupos es presente de forma implícita. Por otro lado, hay libros como el clásico de Budden [2], que aporta ejemplos musicales y de campanología, junto con ejercicios, pero de ninguna manera, como él bien lo dice en el prefacio, pretende ser formal. El capítulo sobre simetrías en la Música en el libro de Benson [1], también ofrece un hermoso panorama de aplicaciones y excelentes ejercicios. Lo que pensamos que distingue el capítulo cuatro de este texto es que pretende tomar algunas de las nociones fundamentales de la Teoría de Grupos, presentadas de manera formal in los primeros tres capítulos y ver como ciertas relaciones musicales preestablecidas pueden ser explicadas en estos términos. También, se emplean algunos de los aspectos de la teoría transformacional y neo-riemannina que han surgido en los últimos 30 años de trabajo académico sobre la Teoría Matemática de la Música, dándoles un trato más formal que lo que permite la naturaleza de otros trabajos sobre el tema. Los ejercicios piden mucho detalle, se dan pistas y las respuestas completas están disponibles.

De hecho, muchos de los ejercicios consisten en completar demostraciones, para asegurar que las sutilezas y detalles son captados, o reforzar cuestiones de notación y simbología que son particulares a la Teoría Matemática de la Música, para que lleguen a ser naturales para aquellos estudiantes interesados en continuar esta línea de estudio.

El capítulo cuatro, de alguna forma, cuenta una historia. La primera sección, Antecedentes Musicales, plantea la relación entre el estudio de las frecuencias de los tonos y el estudio de tonos desde la perspectiva de la estructura. Asimismo, se introduce la notación usada en la Teoría Matemática de la Música y se la relaciona con los conceptos básicos de la Música para él que sólo tiene el conocimiento matemático, o los conceptos básicos de la Teoría de Grupos para los músicos (escala cromática -  $\mathbb{Z}_{12}$ ; definición formal de acorde como un subconjunto del conjunto potencia de  $\mathbb{Z}_{12}$ , etc.) En la segunda sección se toman las transformaciones “T” (transposición) e “I” (inversión) y se definen en términos matemáticos formales, se ofrecen representaciones geométricas, se exploran y se prueban proposiciones sobre las relaciones entre ellos como objetos matemáticos, hasta llegar al “clímax”, donde se demuestra formalmente que el “conjunto” de estas 24 transformaciones de hecho forma un grupo de transformaciones bajo composición. En la tercera sección se hace lo mismo con las transformaciones “P” (paralela), “R” (relativa) y “L” (intercambio de la séptima). Se definen formalmente, se ofrece otro tipo de representación geométrica (el *Tonnetz*, ca-

racterístico de la teoría neo-riemanniana) y, una vez más, se crea un ambiente en que las relaciones entre las transformaciones P, R y L son analizadas y se prueban proposiciones hasta llegar al teorema en que se demuestra que también, vistas como objetos, forman un grupo bajo composición.

En la sección 4.4 se construye un isomorfismo entre los grupos TI y PRL y, en la sección 4.5, se ve que estos dos grupos son duales, eso es, cada uno es el centralizador del otro como subgrupos del grupo de simetrías (permutaciones, que también son transformaciones) del conjunto de los 24 acordes mayores y menores. Aquí, también, se aportan unos ejemplos musicales donde se puede comprobar la dualidad. Como ya se mencionó, este capítulo está basado en [3] y en la tesis de maestría de Janine du Plessis. Una aportación muy importante de du Plessis fue la formalización con sumo detalle de los aspectos mencionados. Esta formalización se traslada al carácter de los ejercicios. A la vez, los ejercicios son razonables y formulados en términos pedagógicos.

## Bibliografía

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